159.

ON SOME INTEGRAL TRANSFORMATIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 4—6.]

SUPPOSE that x, a, b, c and x', a', b', c' have the same anharmonic ratios, or what is the same thing, let these quantities satisfy the equation

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x, & a, & b, & c \\ x', & a', & b', & c' \\ xx', & aa', & bb', & cc' \end{vmatrix} = 0;$$

this equation may be represented under a variety of different forms, which are obtained without difficulty; thus, if for shortness

$$K = a(b'-c')(x'-a') + b\ (c'-a')(x'-b') + c\ (a'-b')(x'-c'),$$

then

$$Kx = -\{bc(b'-c')(x'-a') + ca(c'-a')(x'-b') + ab(a'-b')(x'-c')\},$$

$$K(x-a) = (c-a)(a-b)(b'-c')(x'-a'),$$

$$K(x-b) = (a-b)(b-c)(c'-a')(x'-b'),$$

$$K(x-c) = (b-c)(c-a)(a'-b')(x'-b').$$

Consider x, x' as variables; then

$$K^2\, dx = (b-c)\ (c-a)\ (a-b)(b'-c')(c'-a')(a'-b')\, dx';$$

let, d, d' be any corresponding values of x, x'; then

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ aa', & bb', & cc', & dd' \end{vmatrix} = 0$$

and we have

$$K(x-d) = D(x'-d');$$

where

$$D = (c'-a')(a'-b')(b'-c')\Lambda,$$

and

$$\Lambda = \frac{(a-d)(b-c)}{(a'-d')(b'-c')} = \frac{(b-d)(c-a)}{(b'-d')(c'-a')} = \frac{(c-d)(a-b)}{(c'-d')(a'-b')}.$$

Suppose $\alpha+\beta+\gamma+\delta=-2$; then

$$(x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta\, dx = J(x'-a')^\alpha (x'-b')^\beta (x'-c')^\gamma (x'-d')^\delta\, dx,$$

where

$$J = (b-c)^{\beta+\gamma+1}(c-a)^{\gamma+\alpha+1}(a-b)^{\alpha+\beta+1}(b'-c')^{\alpha+\delta+1}(c'-a')^{\beta+\delta+1}(a'-b')^{\gamma+\delta+1} D^\delta.$$

We may in particular take for a', b', c', d' the systems $b, a, d, c : c, d, a, b$ and d, c, b, a respectively; this gives, writing successively y, z, w instead of x',

$$\begin{aligned}
&(x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta\, dx \\
&= M(y-a)^\beta (y-b)^\alpha (y-c)^\delta (y-d)^\gamma\, dy \\
&= N(z-a)^\gamma (z-b)^\delta (z-c)^\alpha (z-d)^\beta\, dz \\
&= P(w-a)^\delta (w-b)^\gamma (w-c)^\beta (w-d)^\alpha\, dw,
\end{aligned}$$

where

$$\begin{aligned}
M &= -(-)^{\gamma+\delta}(a-c)^{\alpha+\gamma+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(b-d)^{\beta+\delta+1},\\
N &= (-)^{\gamma+\delta}(a-b)^{\alpha+\beta+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(c-d)^{\gamma+\delta+1},\\
P &= (a-b)^{\alpha+\beta+1}(a-c)^{\alpha+\gamma+1}(b-d)^{\beta+\delta+1}(c-d)^{\gamma+\delta+1};
\end{aligned}$$

the relations between the variables x, y, z, w being

$$\begin{aligned}
x &= \frac{(c+d)ab-(a+b)cd-(ab-cd)y}{ab-cd-(a+b-c-d)y}\\
&= \frac{(b+d)ac-(a+c)bd-(ac-bd)z}{ac-bd-(a+c-b-d)z}\\
&= \frac{(b+c)ad-(a+d)bc-(ad-bc)w}{ad-bc-(a+d-b-c)w}:
\end{aligned}$$

these are, in fact, the formulæ in my note, "On an Integral Transformation," *Camb. and Dubl. Math. Jour.* t. III. (1848), p. 286 [62], which was suggested to me by Gudermann's transformation for elliptic functions, (*Crelle*, t. XXIII. (1846), p. 330).

Suppose now that the values of a', b', c', d' are 0, 1, ∞, ζ, we have in this case

$$x = \frac{a(b-c)+c(a-b)y}{(b-c)+(a-b)y},$$

and representing the denominator by K, then

$$\begin{aligned} K(x-a) &= -(a-b)(a-c)\,y, \\ K(x-b) &= \ \ (a-b)(b-c)(1-y), \\ K(x-c) &= \ \ (a-c)(b-c), \\ K(x-d) &= \ \ (a-b)(c-d)(y-\zeta), \end{aligned}$$

where

$$\zeta = \frac{(a-d)(b-c)}{(a-b)(c-d)},$$

and we have

$$K^2\,dx = -(a-b)(a-c)(b-c)\,dy,$$

whence

$$(x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta\,dx = \\ -(-)^\alpha (a-b)^{\alpha+\beta+\delta+1} (a-c)^{\alpha+\gamma+1} (b-c)^{\beta+\gamma+1} (c-d)^\delta\, y^\alpha (1-y)^\beta (y-\zeta)^\delta\,dy.$$

It is easy, by means of this equation, to generalise a remarkable formula given by M. Serret in his memoir, "Sur la Représentation géométrique des Fonctions elliptiques et ultrà-elliptiques," *Liouville*, t. XI. and XII. [1846 and 1847], and *Recueil des Savans étrangers*, t. XI. [1851].(1) In fact, suppose that the indices α, β, γ, δ are integers, and that two of these indices, e.g. γ, δ, are negative, the remaining two indices being positive, then writing $-\gamma$, $-\delta$ instead of γ, δ the integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta\,dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma+\delta=\alpha+\beta+2$, depends on the integral

$$\int \frac{y^\alpha (1-y)^\beta\,dy}{(y-\zeta)^\delta}.$$

Suppose that the fraction under the integral sign is resolved into simple fractions, each of these fractions will be integrable algebraically, except the fraction having for its denominator the simple power $y-\zeta$, the integral of which is a logarithm. The coefficient of this fraction is at once found by writing in the numerator $\zeta+(y-\zeta)$ for y; and expanding in ascending powers of $y-\zeta$ and equating this coefficient to zero, we have

$$\left(\frac{d}{d\zeta}\right)^{\delta-1} \zeta^\alpha (1-\zeta)^\beta = 0,$$

which [observing that $(\gamma-1)+(\delta-1)=\alpha+\beta$] is easily seen to be equivalent to

$$\left(\frac{d}{d\zeta}\right)^{\gamma-1} \zeta^\beta (1-\zeta)^\alpha = 0.$$

[1] M. Serret has reproduced the theorem in his very interesting and instructive treatise, "Cours d'Algèbre supérieure," deuxième édition, Paris, 1854, [quatrième édition, Paris, 1877].

Hence if the function $\zeta = \frac{(a-d)(b-c)}{(a-b)(c-d)}$ satisfy this condition, the indefinite integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta \, dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma + \delta = \alpha + \beta + 2$, will be expressible as a rational algebraical fraction.

It may be noticed, that in the general case, observing that $x = a$, $x = b$ give $y = 0$, $y = 1$, the integral

$$\int_a^b (x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta \, dx$$

depends on

$$\int_0^1 y^\alpha (1-y)^\beta (\zeta - y)^\delta \, dy,$$

or, putting $\zeta = \frac{1}{u}$, upon

$$\int_0^1 y^\alpha (1-y)^\beta (1-uy)^\delta \, dy,$$

which is expressible by means of a hypergeometric series having u for its argument or fourth element.

2, *Stone Buildings, Lincoln's Inn, Feb.* 1854.

160.

ON A THEOREM RELATING TO RECIPROCAL TRIANGLES.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 7—10.]

THE following theorem is, I assume, known; but the analytical demonstration of it depends upon a formula in determinants which is not without interest. The theorem referred to may be thus stated:

"A triangle and its reciprocal are in perspective;" where by the reciprocal of a triangle is meant the triangle the sides of which are the polars of the angles of the first-mentioned triangle with respect to a conic; and triangles are in perspective when the three lines forming the corresponding angles meet in a point, or what is the same thing, when the three points of intersection of the corresponding sides lie in a line.

Let the equation of the conic be

$$x^2+y^2+z^2=0,$$

and take (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ for the coordinates of the angles of the triangle, then if K be the determinant, and (A, B, C) (A', B', C') (A'', B'', C'') the inverse system, i.e. if

$$\begin{aligned} KA\ &= (\beta'\gamma''-\beta''\gamma'), & KB\ &= \gamma'\alpha''-\gamma''\alpha', & KC\ &= \alpha'\beta''-\alpha''\beta', \\ KA' &= (\beta''\gamma-\beta\gamma''), & KB' &= \gamma''\alpha-\gamma\alpha'', & KC' &= \alpha''\beta-\alpha\beta'', \\ KA'' &= (\beta\gamma'-\beta'\gamma), & KB'' &= \gamma\alpha'-\gamma'\alpha, & KC'' &= \alpha\beta'-\alpha'\beta, \end{aligned}$$

equations which may be represented in the notation of matrices by the single equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma' \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}^{-1} = \begin{vmatrix} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{vmatrix},$$

then the equations of the sides of the triangle are

$$\begin{aligned} A\,x + B\,y + C\,z &= 0, \\ A'x + B'y + C'z &= 0, \\ A''x + B''y + C''z &= 0, \end{aligned}$$

and the coordinates of the angles of the reciprocal triangle may be taken to be (A, B, C) (A', B', C') (A'', B'', C''); the equations of the lines joining the corresponding angles of the two triangles are therefore

$$\begin{aligned} (B\gamma - C\beta)\,x + (C\alpha - A\gamma)\,y + (A\beta - B\alpha)\,z &= 0, \\ (B'\gamma' - C'\beta')\,x + (C'\alpha' - A'\gamma')\,y + (A'\beta' - B'\alpha')\,z &= 0, \\ (B''\gamma'' - C''\beta'')\,x + (C''\alpha'' - A''\gamma'')\,y + (A''\beta'' - B''\alpha'')\,z &= 0\,; \end{aligned}$$

the condition that these lines may meet in a point is therefore

$$\begin{vmatrix} B\gamma - C\beta, & C\alpha - A\gamma, & A\beta - B\alpha \\ B'\gamma' - C'\beta', & C'\alpha' - A'\gamma', & A'\beta' - B'\alpha' \\ B''\gamma'' - C''\beta'', & C''\alpha'' - A''\gamma'', & A''\beta'' - B''\alpha'' \end{vmatrix} = 0,$$

an equation which is satisfied identically when A, B, C; A', B', C'; A'', B'', C'' are replaced by their values. To prove this I transform the different quantities which enter into the determinant as follows: putting

$$\begin{aligned} F &= \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'', \\ G &= \alpha''\alpha + \beta''\beta + \gamma''\gamma, \\ H &= \alpha\alpha' + \beta\beta' + \gamma\gamma'; \end{aligned}$$

we have

$$\begin{aligned} K(B\gamma - C\beta) &= \gamma\,(\gamma'\alpha'' - \gamma''\alpha') - \beta\,(\alpha\beta'' - \alpha''\beta') \\ &= \alpha''(\beta\beta' + \gamma\gamma') - \alpha'(\beta\beta'' + \gamma\gamma'') \\ &= \alpha''(\alpha\alpha' + \beta\beta' + \gamma\gamma') - \alpha'(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'') \\ &= \alpha''H - \alpha'G, \\ &\text{\&c.}; \end{aligned}$$

and the equation becomes

$$\begin{vmatrix} \alpha''H - \alpha'G, & \beta''H - \beta'G, & \gamma''H - \gamma'G \\ \alpha F - \alpha''H, & \beta F - \beta''H, & \gamma F - \gamma''H \\ \alpha'G - \alpha F, & \beta'G - \beta F, & \gamma'G - \gamma F \end{vmatrix} = 0.$$

Now the minor $(\beta F - \beta''H)(\gamma'G - \gamma F) - (\gamma F - \gamma''H)(\beta'G - \beta F)$ is equal to

$$GH(\beta'\gamma'' - \beta''\gamma') + HF(\beta''\gamma - \beta\gamma'') + FG(\beta\gamma' - \beta'\gamma),$$

i.e. to

$$K(GHA + HFA' + FGA'');$$

and expressing the other minors in a similar form, the equation to be proved is

$$\left.\begin{array}{r}(GHA+HFA'+FGA'')(B\gamma-C\beta)\\ +(GHB+HFB'+FGB'')(C\alpha-A\gamma)\\ +(GHC+HFC'+FGC'')(A\beta-B\alpha)\end{array}\right\}=0,$$

i. e.

$$HF\begin{vmatrix}A', & B', & C'\\ A, & B, & C\\ \alpha, & \beta, & \gamma\end{vmatrix}+FG\begin{vmatrix}A'', & B'', & C''\\ A, & B, & C\\ \alpha, & \beta, & \gamma\end{vmatrix}=0.$$

The first determinant is

$$-\{\alpha(BC'-B'C)+\beta(CA'-C'A)+\gamma(AB'-A'B)\}=-\frac{1}{K}(\alpha\alpha''+\beta\beta''+\gamma\gamma'')=-\frac{1}{K}G,$$

and the second determinant is

$$\{\alpha(B''C-BC'')+\beta(C''A-CA'')+\gamma(A''B-AB'')\}=\frac{1}{K}(\alpha\alpha'+\beta\beta'+\gamma\gamma')=\frac{1}{K}H,$$

and we have therefore identically

$$HF(-G)+FG(H)=0.$$

The corresponding theorem in geometry of three dimensions is that a tetrahedron and its reciprocal have to each other a certain relation, viz. the four lines joining the corresponding angles are generating lines of a hyperboloid, or, what is the same thing, the four lines of intersection of corresponding faces are generating lines of a hyperboloid. The demonstration would show how the theorem in determinants is to be generalised.

2, *Stone Buildings, Lincoln's Inn, February*, 1855.

161.

A PROBLEM IN PERMUTATIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), p. 79.]

THE game called Mousetrap gives rise to a singular problem in permutations. A set of cards, ace, two, three, &c., say up to thirteen, are arranged in a circle with their faces upwards—you begin at any card, and count one, two, three, &c., and if upon counting suppose the number five, you arrive at the card five, that card is thrown out; and beginning again with the next card, you count one, two, three, &c., throwing out if the case happen a new card as before, and so on until you have counted up to thirteen, without coming to a card which ought to be thrown out. It is easy to see that, whatever the number of the cards is, they may be so arranged as to be all thrown out in the order of their numbers; but that it is not possible in general to arrange the cards so that all the cards, or any specified cards, may be thrown out in a given order. Thus, if all the cards are to be thrown out in the order of their numbers, the arrangements in the case of a single card, two, three, &c. cards, are

1
1 2
1 3 2
1 4 2 3
1 3 2 5 4
1 4 2 5 6 3
1 5 2 7 4 3 6
1 6 2 4 5 3 7 8
&c.

It is required to investigate the general theory.

162.

TWO LETTERS ON CUBIC FORMS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 85—87 and 90—91.]

CHER MONS. HERMITE,

Il y a longtemps que j'ai voulu vous écrire, mais j'en ai été empêché je ne sais comment; j'ai assez à vous dire par rapport aux covariants, mais à présent je vais vous parler des formes cubiques à deux indéterminées. Il me semble que l'on peut simplifier la théorie de Eisenstein, et l'étendre au cas d'un déterminant négatif quelconque, de la manière que voici.

Soit $(a, b, c, d \between x, y)^3$ une forme cubique, je représente par Hessn. $(a, b, c, d \between x, y)^3$ la forme quadratique dérivée $(ac - b^2, \tfrac{1}{2}(ad - bc), bd - c^2 \between x, y)^2$. Cela étant, soit (A, B, C) une forme représentative (réduite et proprement primitive) au déterminant $-D$; à moins que $(A, B, C)^2 = (A, -B, C)$, c'est-à-dire, à moins que (A, B, C) ne soit une forme laquelle par sa triplication produit la forme principale, il n'existe pas de forme cubique (a, b, c, d) telle que $-$ Hessn. $(a, b, c, d \between x, y)^3 = (A, B, C \between x, y)^2$, ou, si l'on veut, telle que $b^2 - ac = A$, $bc - ad = 2B$, $c^2 - bd = C$; mais en supposant que l'on ait $(A, B, C)^2 = (A, -B, C)$ on peut trouver une seule forme cubique qui satisfait à l'équation dont il s'agit. J'écarte, cela va sans dire, l'une ou l'autre des deux formes (a, b, c, d) et $(-a, -b, -c, -d)$.

En effet on a identiquement

$$(b^2 - ac, -\tfrac{1}{2}(bc - ad), c^2 - bd \between bxx' + cxy' + cx'y + dyy', axx' + bxy' + bx'y + cyy')^2$$
$$= (b^2 - ac, \tfrac{1}{2}(bc - ad), c^2 - bd \between x, y)^2 \times (b^2 - ac, \tfrac{1}{2}(bc - ad), c^2 - bd \between x', y')^2,$$

donc, en supposant que $b^2 - ac = A$, $bc - ad = 2B$, $c^2 - bd = C$, il s'ensuit que $(A, B, C)^2 = (A, -B, C)$.

Je suppose donc $(A,\ B,\ C)^2 = (A,\ -B,\ C)$, et je dis qu'il ne peut pas y avoir deux formes cubiques, $(a,\ b,\ c,\ d)$ et $(a_{,},\ b_{,},\ c_{,},\ d_{,})$, qui aient la propriété dont il s'agit; car, en écrivant

$$\left.\begin{aligned}\xi &= bxx' + cxy' + cx'y + dyy',\\ \eta &= axx' + bxy' + bx'y + cyy',\end{aligned}\right\}\qquad \left.\begin{aligned}\xi_{,} &= b_{,}xx' + c_{,}xy' + c_{,}x'y + d_{,}yy',\\ \eta_{,} &= a_{,}xx' + b_{,}xy' + b_{,}x'y + c_{,}yy',\end{aligned}\right\}$$

on trouverait

$$(A,\ -B,\ C\between \xi,\ \eta)^2 = (A,\ -B,\ C\between \xi_{,},\ \eta_{,})^2,$$

ce qui implique d'abord que $\xi_{,}$, $\eta_{,}$ soient des fonctions linéaires de ξ, η. Mais $(A, -B, C)$ étant une forme réduite et proprement primitive au déterminant $-D$, il n'existe pas de transformation de la forme quadratique en elle-même, hormis $\xi_{,} = \xi$, $\eta_{,} = \eta$. Le cas $D = 1$ doit se traiter à part; dans ce cas particulier il n'y a que la forme cubique $(0,\ 1,\ 0,\ 1)$. Donc &c. Enfin, si $(A,\ B,\ C)^2 = (A,\ -B,\ C)$, il existe une forme cubique $(a,\ b,\ c,\ d)$ telle que $b^2 - ac = A$, &c.; car en cherchant par la méthode de Gauss les valeurs des coefficients $p,\ p',\ p'',\ p'''$ et $q,\ q',\ q'',\ q'''$ qui donnent cette transformation, on obtient d'abord $p' = p''$, $q' = q''$. On peut donc représenter ces coefficients par $b_{,},\ c_{,},\ c_{,},\ d_{,}$; $a,\ b,\ b,\ c$; savoir, on peut trouver $a,\ b,\ c,\ b_{,},\ c_{,},\ d_{,}$ de manière que

$$\begin{aligned}(A,\ -B,\ C\between b_{,}xx' + c_{,}xy' + c_{,}x'y + d_{,}yy',\quad & axx' + bxy' + bx'y + cyy')^2\\ &= (A,\ B,\ C\between x,\ y)^2 . (A,\ B,\ C\between x',\ y')^2.\end{aligned}$$

Cela étant, les équations de Gauss donnent

$$\begin{aligned} A &= bb_{,} - ac_{,}, & A &= b^2 \ - ac,\\ 2B &= cb_{,} - ad_{,}, & -2B &= cb_{,} + ad_{,} - 2bc_{,},\\ C &= cc_{,} - bd_{,}, & C &= c_{,}^2 \ - b_{,}d_{,},\end{aligned}$$

et de là on obtient

$$\begin{aligned} b\ (b - b_{,}) - a\ (c - c_{,}) &= 0,\\ c\ (b - b_{,}) - b\ (c - c') &= 0,\\ c_{,}(b - b_{,}) - b_{,}(c - c_{,}) &= 0,\\ d_{,}(b - b_{,}) - c_{,}(c - c_{,}) &= 0\,;\end{aligned}$$

c'est-à-dire, ou $\frac{a}{b} = \frac{b}{c} = \frac{b_{,}}{c_{,}} = \frac{c_{,}}{d_{,}}$, ce qui n'est pas vrai (car cela donnerait $A = 0$, $B = 0$, $C = 0$), ou $b - b_{,} = 0$, $c - c_{,} = 0$. Donc $b_{,} = b$, $c_{,} = c$; et en écrivant d au lieu de $d_{,}$, on voit que l'équation de transformation devient

$$\begin{aligned}(A,\ -B,\ C\between bxx' + cxy' + cx'y + dyy',\quad & axx' + bxy' + bx'y + cyy')^2\\ &= (A,\ B,\ C\between x,\ y)^2 . (A,\ B,\ C\between x,\ y')^2,\end{aligned}$$

où

$$A = b^2 - ac,\quad 2B = bc - ad,\quad C = c^2 - bd.\quad \text{C. q. f. à d.}$$