101.

NOTES ON LAGRANGE'S THEOREM.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VI. (1851), pp. 37—45.]

I.

If in the ordinary form of Lagrange's theorem we write $(x+a)$ for x, it becomes

$$x = hf(a+x),$$

$$F(a+x) = Fa + \frac{h}{1}F'afa + \&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

It follows that the equation

$$F(a+x) = Fa + \frac{1}{1}\frac{x}{f(a+x)}(F'afa) + , \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

must reduce itself to an identity when the two sides are expanded in powers of x; or writing for shortness F, f instead of Fa, fa, and δ for $\frac{d}{da}$, we must have

$$\frac{1}{[r]^r}\delta^r F = S\left\{\frac{1}{[p]^p}\delta^{p-1}(\delta F\,.\,f^p)\frac{1}{[r-p]^{r-1}}\delta^{r-p}f^{-p}\right\}, \ldots\ldots\ldots\ldots(3)$$

(where p extends from 0 to r). Or what comes to the same,

$$\frac{1}{[r]^r}\delta^r F = S\left\{\frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}\,[s-1]^{s-1}}\delta^{p-s}f^p\,.\,\delta^{r-p}f^{-p}\,.\,\delta^s F\right\}, \ldots\ldots(4)$$

where s extends from 0 to $(r-p)$. The terms on the two sides which involve $\delta^r F$ are immediately seen to be equal; the coefficients of the remaining terms $\delta^s F$ on the second side must vanish, or we must have

$$S\left\{\frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}}(\delta^{p-s}f^p)(\delta^{r-p}f^{-p})\right\} = 0, \ldots\ldots\ldots\ldots (5)$$

(s being less than r). Or in a somewhat more convenient form, writing p, q and k for $p-s$, $r-p$ and $r-s$,

$$S\left\{\frac{1}{(p+s)\,[p]^p\,[s]^s}(\delta^p f^{p+s})\,(\delta^q f^{-p-s})\right\}=0, \qquad (6)$$

where s is constant and p and q vary subject to $p+q=k$, k being a given constant different from zero (in the case where $k=0$, the series reduces itself to the single term $\frac{1}{s}$). The direct proof of this theorem will be given presently.

II.

The following symbolical form of Lagrange's theorem was given by me in the *Mathematical Journal*, vol. III. [1843], pp. 283—286, [8].

If

$$x=a+hfx, \qquad (7)$$

then

$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}F'a\,e^{hfa}.$$

Suppose $fx=\phi\,(b+k\psi x)$, or $x=a+h\phi\,(b+k\psi x)$, then

$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}F'a\,e^{h\phi\,(b+k\psi a)}.$$

But

$$e^{h\phi\,(b+k\psi a)}=\left(\frac{d}{db}\right)^{k\frac{d}{dk}}e^{h\phi b+k\psi a}.$$

(In fact the two general terms

$$\{\phi\,(b+k\psi a)\}^m \text{ and } \left(\frac{d}{db}\right)^{k\frac{k}{dk}}e^{k\psi a}\,(\phi b)^m,$$

of which the former reduces itself to $e^{k\psi a\frac{d}{db}}(\phi b)^m$, are equal on account of the equivalence of the symbols

$$e^{k\psi a\frac{d}{db}} \text{ and } \left(\frac{d}{db}\right)^{k\frac{d}{dk}}e^{k\psi a}).$$

Hence

$$x=a+h\phi\,(b+k\psi x), \qquad (8)$$

$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}\left(\frac{d}{db}\right)^{k\frac{d}{dk}}F'ae^h\phi^{b+k\psi a};$$

and the coefficient of $h^m k^n$ is

$$\frac{1}{[m]^m\,[n]^n}\left(\frac{d}{da}\right)^{m-1}F'a\,(\psi a)^n.\left(\frac{d}{db}\right)^n(\phi b)^n.$$

A similar formula evidently applies to the case of any finite number of functions ϕ, ψ, &c.: in the case of an infinite number we have

$$F\left(a+h\phi\left(b+k\psi\left(c+l\chi\left(d+\ldots\right)=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}\left(\frac{d}{db}\right)^{k\frac{d}{dk}}\left(\frac{d}{dc}\right)^{l\frac{d}{dl}}\ldots F'a\,e^{h\phi b+k\psi c+l\chi d+\ldots};\ \ldots(9)$$

or the coefficient of $h^m k^n l^p \ldots$ is

$$\frac{1}{[m]^m[n]^n[p]^p\ldots}\left(\frac{d}{da}\right)^m Fa\,.\left(\frac{d}{db}\right)^n(\phi b)^m\,.\left(\frac{d}{dc}\right)^p(\psi c)^r\ldots$$

the last of the series m, n, p...... being always zero; e.g. in the coefficient of $h^m k^n$, account must be had of the factor $\left(\frac{d}{dc}\right)^p(\psi c)^n$ or $(\psi c)^n$. The above form is readily proved independently by Taylor's theorem, without the assistance of Lagrange's. If in it we write $h=k$, &c., $a=b=$ &c., and $\phi=\psi=$ &c. $=f$, we have $F(a+hf(a+hf(a+\ldots)=Fx$, where $x=a+hfx$. Hence, comparing the coefficient of h^s with that given by Lagrange's theorem,

$$\frac{1}{[s]^s}\,\delta^{s-1}(\delta F\,.f^s)=S\left\{\frac{1}{[m]^m[n]^n[p]^p\ldots}\,\delta^m F\,.\,\delta^n f^m\,.\,\delta^p f^r\ldots\right\},\ \ldots\ldots\ (10)$$

where $m+n+$ &c. $=s$, and as before Fa, fa, $\frac{d}{da}$ have been replaced by F, f, δ. By comparing the coefficients of $\delta^m F$,

$$\frac{1}{[t]^t}\,\frac{s-t}{s}\,\delta^t f^s=\Sigma\left\{\frac{1}{[n]^n[p]^p\ldots}(\delta^n f^{s-t})(\delta^p f^n)\ldots\right\},\ \ldots\ldots\ldots\ldots\ldots(11)$$

where $n+p+\ldots=t$, the last of the series n, p... always vanishing. The formula (10) deduced, as above mentioned, from Taylor's theorem, and the subsequent formula (11) with an independent demonstration of it, not I believe materially different from that which will presently be given, are to be found in a memoir by M. Collins (volume II. (1833) of the Memoirs of the Academy of St Petersburg), who appears to have made very extensive researches in the theory of developments as connected with the combinatorial analysis.

III.

To demonstrate the formula (6), consider, in the first place, the expression

$$S\,\frac{\phi p}{[p]^p[q]^q}\{(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta})\},$$

where $p+q=k$. Since

$$\frac{1}{[p]^p[q]^q}=\frac{1}{k}\left(\frac{1}{[p-1]^{p-1}[q]^q}+\frac{1}{[p]^p[q-1]^{q-1}}\right),$$

this is immediately transformed into

$$\frac{1}{k} S\phi p \left\{\frac{p+s}{[p-1]^{p-1}[q]^q}(\delta^{p-1}f^{p+s-1}\delta f)(\delta^q f^{-p-s-\theta}) - \frac{(p+s+\theta)}{[p]^p[q-1]^{q-1}}(\delta^p f^{p+s})(\delta^{q-1}.f^{-p-s-\theta-1}\delta f)\right.$$

$$= \frac{1}{k} S \frac{1}{[p]^p[q]^q}\{\phi(p+1)(p+s+1)(\delta^p.f^{p+s}\delta f)(\delta^q f^{-p-s-\theta-1})$$

$$-\phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^q.f^{-p-s-\theta-1}\delta f)\},$$

in which last expression $p+q=(\rho-1)$. Of this, after separating the factor δf, the general term is

$$\frac{1}{k}\frac{1}{[\alpha]^\alpha}\delta^{\alpha+1}f.S\left\{\frac{1}{[p-\alpha]^{p-\alpha}[q]^q}\phi(p+1)(p+s+1)(\delta^{p-\alpha}f^{p+s})(\delta^q f^{-p-s-\theta-1})\right.$$

$$\left.-\frac{1}{[p]^p[q-\alpha]^{q-\alpha}}\phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^{q-\alpha}f^{-p-s-\theta-1})\right\},$$

equivalent to

$$\frac{1}{k}\frac{1}{[\alpha]^\alpha}\delta^{\alpha+1}f.S\frac{1}{[p]^p[q]^q}\{\phi(p+\alpha+1)(p+s+\alpha+1)(\delta^p f^{p+s+\alpha})(\delta^q f^{-p-s-\alpha-\theta-1})$$

$$-\phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta-1})\},$$

in which last expression $p+q=k-\alpha-1$. By repeating the reduction j times, the general term becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\;\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots$$

$$\times S\frac{1}{[p^p][q]^q}\Sigma\{(-)^{j-j'}\phi(p+\alpha+\beta\ldots+j')[p+s+\alpha+\beta\ldots+j']^{j'}$$

$$\times[p+s+\theta+\alpha+\beta\ldots+j-1]^{j-j'}(\delta^p f^{p+s+\alpha+\beta\ldots})(\delta^q f^{-p-s-\theta-j-\alpha-\beta\ldots})\},$$

where the sums $\alpha+\beta\ldots$ contain j' terms, j' being less than j or equal to it, and Σ extends to all combinations of the quantities α, $\beta\ldots$ taken j' and j' together (so that the summation contains 2^j terms). Also $p+q=k-\alpha-\beta\ldots$ (j terms) $-j$, and the products $k(k-\alpha-1)(k-\alpha-\beta-2)\ldots$ and $[\alpha]^\alpha[\beta]^\beta,\ldots\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots$ contain each of them j terms. Suppose the reduction continued until $k-\alpha-\beta\ldots$ (j terms) $-j=0$, then the only values of p, q are $p=0$, $q=0$; and the general term of

$$S\frac{\phi p}{[p]^p[q]^q}\{(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\;\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots f^{-j-\theta}$$

$$\times\Sigma\{(-)^{j-j}\phi(\alpha+\beta\ldots+j')[s+\alpha+\beta\ldots+j']^{j'}[s+\theta+\alpha+\beta\ldots+j-1]^{j-j'}\}.$$

If $\theta=0$, the general term reduces itself to

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\,\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\,\delta^{\alpha+1}f.\,\delta^{\beta+1}f\ldots f^{-j}.$$

$$\Sigma\{(-)^{j-j'}(s+\alpha+\beta\ldots+j')\,\phi(\alpha+\beta\ldots+j')\,[s+\alpha+\beta\ldots+j-1]^{j-1}\};$$

whence finally, if $\phi p=\dfrac{1}{p+s}$, the general term of

$$S\frac{1}{(p+s)[p]^p[q]^q}\{(\delta^p f^{p+s})(\delta^q f^{-p-s})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\cdot\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\,\delta^{\alpha+1}f.\,\delta^{\beta+1}f\ldots f^{-j}\,\Sigma\{(-)^{j-j'}[s+\alpha+\beta\ldots+j-1]^{j-1}\};$$

and it is readily shown that the sum contained in this formula vanishes, which proves the equation in question.

IV.

The demonstration of the equation (11) is much simpler. We have

$$\delta^{t-1}(f^{s-1}\delta f)=\Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}\,\delta^{n-1}(f^{s-t-1}\delta f).(\delta^{t-n}f^t)\right\},$$

that is,

$$\delta^t f^s=\frac{s}{s-t}\,\Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}\,(\delta^n f^{s-t})(\delta^{t-n}f^t)\right\},$$

where n extends from $n=1$ to $n=t$. Similarly

$$\delta^{t-n}f^t=\frac{t}{n}\,\Sigma\left\{\frac{[t-n-1]^{p-1}}{[p-1]^{p-1}}\,(\delta^p f^n)(\delta^{t-n-p}f^{t-n})\right\},$$

$$\delta^{t-n-p}f^{t-n}=\frac{t-n}{p}\,\Sigma\left\{\frac{[t-n-p-1]^{q-1}}{[q-1]^{q-1}}\,(\delta^q f^p)(\delta^{t-n-p-q}f^{t-n-p})\right\},$$

&c.

Hence, substituting successively, and putting $t-n-p-q=r$, &c.,

$$\delta^t f^s=\frac{s}{s-t}\,\Sigma\,\frac{[t]^{t-r-1}}{[n]^n[p]^p(q+r)[q-1]^{q-1}}\,(\delta^n f^{s-t})(\delta^q f^p)(\delta^r f^{q+r})\Big\},$$

&c.; and the last of these corresponding to a zero value of the last of the quantities $n, p, q\ldots$ is evidently the required equation (11).

V.

The formula (18) in my paper on Lagrange's theorem (before referred to) is incorrect. I propose at present, after giving the proper form of the formula in question, to develope the result of the substitution indicated at the conclusion of the paper. It will be convenient to call to mind the general theorem, that when any number of variables $x, y, z\ldots$ are connected with as many other variables $u, v, w\ldots$ by the same number of equations (so that the variables of each set may be considered as functions of those of the other set) the quotient of the expressions $dxdy\ldots$ and $dudv\ldots$ is equal to the quotient of two determinants formed with the functions which equated to zero express the relations between the two sets of variables; the former with the differential coefficients of these functions with respect to $u, v\ldots$, the latter with the differential coefficients with respect to $x, y\ldots$. Consequently the notation $\dfrac{dxdy\ldots}{dudv\ldots}$ may be considered as representing the quotient of these determinants. This being premised, if we write

$$x-u-h\theta\,(x,\ y\ \ldots)=0,$$
$$y-v-k\phi\,(x,\ y\ \ldots)=0,$$

then the formula in question is

$$F(x,\ y\ \ldots)\frac{dxdy\ldots}{dudv\ldots}=\delta_u{}^{h\delta_h}\delta_v{}^{k\delta_k}\ldots e^{h\theta+k\phi}\cdot F,$$

if for shortness the letters θ, $\phi, \ldots$, F denote what the corresponding functions become when $u, v, \ldots$ are substituted for x, $y, \ldots$. Let $\dfrac{1}{\Delta}$ denote the value which $\dfrac{dxdy\ldots}{dudv\ldots}$, considered as a function of x, $y\ldots$, assumes when these variables are changed into $u, v, \ldots$, we have

$$\nabla=\begin{vmatrix} 1-h\delta_u\theta, & -h\delta_v\theta\ldots \\ -k\delta_u\phi, & 1-k\delta_v\phi\ldots \\ \vdots & \end{vmatrix}$$

By changing the function F, we obtain

$$F(x,\ y\ \ldots)=\delta_u{}^{h\delta_h}\delta_v{}^{kd_k}\ldots e^{h\theta+k\phi\ldots}\,F\,\nabla\,;$$

where, however, it must be remembered that thé $h, k, \ldots$, in so far as they enter into the function ∇, are not affected by the symbols $h\delta_h$, $k\delta_k, \ldots$ In order that we may consider them to be so affected, it is necessary in the function ∇ to replace h, k, &c. by $\dfrac{h}{\delta_u}$, $\dfrac{k}{\delta_v}$, &c. Also, after this is done, observing that the symbols $h\delta_u\theta$, $h\delta_v\theta\ldots$ affect a function $e^{h\theta+k\phi+\ldots}F$, the symbols $h\delta_u\theta$, $h\delta_v\theta,\ldots$ may be replaced by $\delta_u{}^\theta$, $\delta_v{}^\theta, \ldots$, where the θ is not an index, but an affix denoting that the differentiation is only to be performed with respect to u, $v\ldots$ so far as these variables respectively enter into the function θ. Transforming the other lines of the determinant in the same manner, and taking out from $\delta_u{}^{h\delta_h}\delta_v{}^{k\delta_k}\ldots$ the factor $\delta_u\delta_v\ldots$ in order to multiply this last factor into the determinant, we obtain

$$F(x,\ y\ \ldots)=\delta_u{}^{h\delta_h-1}\,\delta_v{}^{k\delta_k-1}\ldots e^{h\theta+k\phi\ldots}\,F\,\square\,;$$

where

$$\square=\begin{vmatrix} \delta_u-\delta_u{}^\theta, & -\delta_u{}^\phi,\ldots \\ -\delta_v{}^\theta, & \delta_v-\delta_v{}^\phi, \\ \vdots & \end{vmatrix}$$

in which expression δ_u, δ_v ... are to be replaced by

$$\delta_u{}^F + \delta_u{}^\theta + \delta_u{}^\phi \ldots \delta_v{}^F + \delta_v{}^\theta + \delta_v{}^\phi \ldots .$$

The complete expansion is easily arrived at by induction, and the form is somewhat singular. In the case of a single variable u we have $\square = \delta_u{}^F$, in the case of two variables, $\square = \delta_u{}^F\delta_v{}^F + \delta_u{}^F\delta_v{}^\theta + \delta_u{}^\phi\delta_v{}^F$. Or writing down only the affixes, in the case of a single variable we have F; in the case of two variables FF, $F\theta$, ϕF; and in the case of three variables FFF, ϕFF, χFF, $F\chi F$, $F\theta F$, $FF\theta$, $FF\phi$, $F\theta\theta$, $F\theta\phi$, $F\chi\theta$, $\phi F\phi$, $\chi F\phi$, $\phi F\theta$, $\chi\chi F$, $\phi\chi F$, $\chi\theta F$; where it will be observed that θ never occurs in the first place, nor ϕ in the second place, nor θ, ϕ (in any order) in the first and second places, &c., nor θ, ϕ, χ (in any order) in the first, second, and third places. And the same property holds in the general case for each letter and binary, ternary, &c. combination, and for the entire system of letters, and the system of affixes contains every possible combination of letters not excluded by the rule just given. Thus in the case of two letters, forming the system of affixes FF, $F\theta$, ϕF, θF, $F\phi$, $\theta\phi$, $\phi\theta$, the last four are excluded, the first three of them by containing θ in the first place or ϕ in the second place, the last by containing ϕ, θ in the first and second places: and there remains only the terms FF, $F\theta$, ϕF forming the system given above. Substituting the expanded value of $\square$ in the expression for $F(x, y \ldots)$, the equation may either be permitted to remain in the form which it thus assumes, or we may, in order to obtain the finally reduced form, after expanding the powers of h, $k \ldots$, connect the symbols $\delta_u{}^\theta$, $\delta_u{}^\phi \ldots \delta_u{}^F$, &c. with the corresponding functions θ, $\phi \ldots F$, and then omit the affixes; thus, in particular, in the case of a single variable the general term of Fx is

$$\frac{h^p}{[p]^p}\,\delta_u{}^{p-1}(\theta^p\delta_u F),$$

(the ordinary form of Lagrange's theorem). In the case of two letters the general term of $F(x, y)$ is

$$\frac{h^p k^q}{[p]^p\,[q]^q}\,\delta_u{}^{p-1}\delta_v{}^{q-1}\{\theta^p\phi^q\delta_u\delta_v F + \phi^q\delta_v\theta^p\delta_u F + \theta^p\delta_u\phi^q\delta^v F\}$$

(see the *Mécanique Céleste*, [Ed. 1, 1798] t. I. p. 176). In the case of three variables, the general term is

$$\frac{h^p k^q l^r}{[p]^p\,[q]^q\,[r]^r}\,\delta_u{}^{p-1}\delta_v{}^{q-1}\delta_w{}^{r-1}\{\theta^p\phi^q\chi^r\delta_u\delta_v\delta_w F + \ldots\},$$

the sixteen terms within the { } being found by comparing the product $\delta_u\delta_v\delta_w$ with the system FFF, ϕFF, &c., given above, and then connecting each symbol of differentiation with the function corresponding to the affix. Thus in the first term the δ_u, δ_v, δ_w, each affect the F, in the second term the δ_u affects ϕ^q, and the δ_v and δ_w each affect the F, and so on for the remaining terms. The form is of course deducible from Laplace's general theorem, and the actual development of it is given in Laplace's Memoir in the *Hist. de l'Acad.* 1777. I quote from a memoir by Jacobi which I take this opportunity of referring to, "De resolutione equationum per series infinitas," *Crelle*, t. VI. [1830], pp. 257—286, founded on a preceding memoir, "Exercitatio algebraica circa discerptionem singularem fractionum quæ plures variabiles involvunt," t. V. [1830], pp. 344—364.

Stone Buildings, April 6, 1850.

102.

ON A DOUBLE INFINITE SERIES.

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THE following completely paradoxical investigation of the properties of the function Γ (which I have been in possession of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series.

Let $\Sigma_r \phi r$ denote the sum of the values of ϕr for all integer values of r from $-\infty$ to ∞. Then writing

$$u = \Sigma_r [n-1]^r x^{n-1-r}, \quad \ldots\ldots (1)$$

(where n is any number whatever), we have immediately

$$\frac{du}{dx} = \Sigma_r [n-1]^{r+1} x^{n-2-r} = \Sigma_r [n-1]^r x^{n-1-r} = u;$$

that is, $\dfrac{du}{dx} = u$, or $u = C_n e^x$,

(the constant of integration being of course in general a function of n). Hence

$$C_n e^x = \Sigma_r [n-1]^r x^{n-1-r}; \quad \ldots\ldots (2)$$

or e^x is expanded in general in a *doubly infinite necessarily divergent series of fractional powers of* x, (which resolves itself however in the case of n a positive or negative integer, into the ordinary singly infinite series, the value of C_n in this case being immediately seen to be Γn).

The equation (2) in its general form is to be considered as a definition of the function C_n. We deduce from it

$$\Sigma_r \; [n-1]^r \; (ax)^{n-1-r} = C_n e^{ax},$$
$$\Sigma_{r'} \; [n'-1]^{r'} (ax')^{n-1-r'} = C_{n'} e^{ax'};$$

and also

$$\Sigma_k [n+n' \dots -1]^k \{a(x+x' \dots)\}^{n+n' \dots -1-k} = C_{n+n' \dots} e^{a(x+x' \dots)}.$$

Multiplying the first set of series, and comparing with this last,

$$C_{n+n' \dots} \Sigma_{r, r' \dots} [n-1]^r [n'-1]^{r'} \dots x^{n-1-r} x'^{n'-1-r'} \dots$$
$$= C_n C_{n'} \dots [n+n' \dots -1]^k (x+x' \dots)^{n+n' \dots -1-k}, \dots\dots\dots\dots (3)$$

(where r, r' denote any positive or negative integer numbers satisfying $r+r'+\dots = k+1-p$, p being the number of terms in the series n, n', ...). This equation constitutes a multinomial theorem of a class analogous to that of the exponential theorem contained in the equation (2).

In particular

$$C_{n+n' \dots} \Sigma_{r, r' \dots} [n-1]^r [n'-1]^{r'} \dots = C_n C_{n'} \dots [n+n' \dots -1]^k p^{n+n' \dots -1-k}, \dots\dots\dots (4)$$

and if $p=2$, writing also m, n for n, n', and $k-1-r$ for r',

$$C_{m+n} \Sigma_r [m-1]^r [n-1]^{k-1-r} = C_m C_n [m+n-1]^k \, 2^{m+n-1-k}, \dots\dots\dots\dots\dots\dots (5)$$

or putting $k=0$ and dividing,

$$C_m C_n \div C_{m+n} = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}. \dots\dots\dots\dots\dots (6)$$

Now the series on the second side of this equation is easily seen to be convergent (at least for positive values of m, n). To determine its value write

$$\mathsf{F}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx;$$

then

$$\mathsf{F}(m, n) = \int_0^{\frac{1}{2}} x^{m-1} (1-x)^{n-1} \, dx + \int_0^{\frac{1}{2}} x^{n-1} (1-x)^{m-1} \, dx;$$

and by successive integrations by parts, the first of these integrals is reducible to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from -1 to $-\infty$ inclusively, and the second to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from 0 to ∞; hence

$$\mathsf{F}(m, n) = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r},$$

or

$$C_m C_n \div C_{m+n} = \mathsf{F}(m, n), \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

which proves the identity of C_m with the function $\Gamma(m)$. {Substituting in two of the preceding equations, we have

$$\Gamma n \Gamma n' \ldots \div \Gamma(n + n' \ldots) = \frac{1}{[n + n' \ldots - 1]^k \, p^{n+n' \ldots -1-k}} \Sigma_{r, r' \ldots} [n-1]^r \, [n'-1]^{r'} \ldots, \quad \ldots (8)$$

(where, as before, p denotes the number of terms in the series $n, n', \ldots$ and $r + r' + \ldots = k + 1 - p$), the first side of which equation is, it is well known, reducible to a multiple definite integral by means of a theorem of M. Dirichlet's. And

$$\mathsf{F}(m, n) = \frac{1}{[m + n - 1]^k \, 2^{m+n-1-k}} \Sigma_r [m-1]^r \, [n-1]^{k-1-r}, \quad \ldots\ldots\ldots\ldots(9)$$

where r extends from $-\infty$ to $+\infty$, and k is arbitrary. By giving large negative values to this quantity, very convergent series may be obtained for the calculation of $\mathsf{F}(m, n)$}.