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1.

ON A THEOREM IN THE GEOMETRY OF POSITION.

[From the *Cambridge Mathematical Journal*, vol. II. (1841), pp. 267—271.]

We propose to apply the following (new?) theorem to the solution of two problems in Analytical Geometry.

Let the symbols

$$|\alpha|, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ \&c.}$$

denote the quantities

$$\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma, \text{ \&c.}$$

(the law of whose formation is tolerably well known, but may be thus expressed,

$$|\alpha| = \alpha, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} = \alpha\,|\beta'| - \alpha'\,|\beta|,$$

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \alpha \begin{vmatrix} \beta', & \gamma' \\ \beta'', & \gamma'' \end{vmatrix} + \alpha' \begin{vmatrix} \beta'', & \gamma'' \\ \beta, & \gamma \end{vmatrix} + \alpha'' \begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \text{ \&c.}$$

the signs $+$ being used when the number of terms in the side of the square is odd, and $+$ and $-$ alternately when it is even.)

Then the theorem in question is

$$\begin{vmatrix} \rho\alpha + \sigma\beta + \tau\gamma.., & \rho\alpha' + \sigma\beta' + \tau\gamma'.., & \rho\alpha'' + \sigma\beta'' + \tau\gamma''.. \\ \rho'\alpha + \sigma'\beta + \tau'\gamma.., & \rho'\alpha' + \sigma'\beta' + \tau'\gamma'.., & \rho'\alpha'' + \sigma'\beta'' + \tau'\gamma''.. \\ \rho''\alpha + \sigma''\beta + \tau''\gamma.., & \rho''\alpha' + \sigma''\beta' + \tau''\gamma'.., & \rho''\alpha'' + \sigma''\beta'' + \tau''\gamma''.. \\ \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} \rho, & \sigma, & \tau.. \\ \rho', & \sigma', & \tau'.. \\ \rho'', & \sigma'', & \tau''.. \\ \vdots & \vdots & \vdots \end{vmatrix} \begin{vmatrix} \alpha'', & \beta'', & \gamma''.. \\ \alpha', & \beta', & \gamma'.. \\ \alpha, & \beta, & \gamma.. \\ \vdots & \vdots & \vdots \end{vmatrix}$$

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Excerpt
More information

(This theorem admits of a generalisation which we shall not have occasion to make use of, and which therefore we may notice at another opportunity.)

To find the relation that exists between the distances of five points in space.

We have, in general, whatever x_1, y_1, z_1, w_1, &c. denote

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2+w_1^2, & -2x_1, & -2y_1, & -2z_1, & -2w_1, & 1 \\ x_2^2+y_2^2+z_2^2+w_2^2, & -2x_2, & -2y_2, & -2z_2, & -2w_2, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2+w_5^2, & -2x_5, & -2y_5, & -2z_5, & -2w_5, & 1 \\ 1, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & w_1, & x_1^2+y_1^2+z_1^2+w_1^2 \\ 1, & x_2, & y_2, & z_2, & w_2, & x_2^2+y_2^2+z_2^2+w_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & x_5, & y_5, & z_5, & w_5, & x_5^2+y_5^2+z_5^2+w_5^2 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \overline{x_1-x_1}^2+\overline{y_1-y_1}^2+\overline{z_1-z_1}^2+\overline{w_1-w_1}^2, & \overline{x_1-x_2}^2+\ldots, & \overline{x_1-x_3}^2+\ldots, & \overline{x_1-x_4}^2+\ldots, & \overline{x_1-x_5}^2+\ldots, & 1 \\ \overline{x_2-x_1}^2+\ldots, & \overline{x_2-x_2}^2+\ldots, & \overline{x_2-x_3}^2+\ldots, & \overline{x_2-x_4}^2+\ldots, & \overline{x_2-x_5}^2+\ldots, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{x_5-x_1}^2+\ldots, & \overline{x_5-x_2}^2+\ldots, & \overline{x_5-x_3}^2+\ldots, & \overline{x_5-x_4}^2+\ldots, & \overline{x_5-x_5}^2+\ldots, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

Putting the w's equal to 0, each factor of the first side of the equation vanishes, and therefore in this case the second side of the equation becomes equal to zero. Hence x_1, y_1, z_1, x_2, y_2, z_2, &c. being the coordinates of the points 1, 2, &c. situated arbitrarily in space, and $\overline{12}^2$, $\overline{13}^2$, &c. denoting the squares of the distances between these points, we have immediately the required relation

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

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Excerpt
More information

which is easily expanded, though from the mere number of terms the process is somewhat long.

Precisely the same investigation is applicable to the case of four points in a plane, or three points in a straight line. Thus the former gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The latter gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0;$$

or expanding,

$$\overline{12}^4 + \overline{13}^4 + \overline{23}^4 - 2\,.\,\overline{12}^2\,\overline{13}^2 - 2\,.\,\overline{13}^2\,\overline{23}^2 - 2\,.\,\overline{12}^2\,\overline{23}^2 = 0;$$

which may be derived immediately from the equation

$$\pm\,\overline{12} \pm \overline{13} = \pm\,\overline{23},$$

and is the simplest form under which this equation, cleared of the ambiguous signs, can be put.

(The above result may be deduced so elegantly from the general theory of elimination, that notwithstanding its simplicity it is perhaps worth mentioning.)

Let $$x_{\prime\prime} - x_{\prime\prime\prime} = \alpha,\quad x_{\prime\prime\prime} - x_{\prime} = \beta,\quad \overline{x_{\prime} - x_{\prime\prime}} = \gamma;$$

then $$\overline{12}^2 = \gamma^2,\quad \overline{23}^2 = \alpha^2,\quad \overline{31}^2 = \beta^2,\quad \text{and}\quad \alpha + \beta + \gamma = 0;$$

from which α, β, γ are to be eliminated. Multiplying the last equation by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and reducing by the three first,

$$\begin{aligned} 0\,.\,\alpha + \overline{12}^2\,.\,\beta + \overline{31}^2\,.\,\gamma + \quad \alpha\beta\gamma &= 0, \\ \overline{12}^2\,.\,\alpha + \ 0\,.\,\beta + \overline{23}^2\,.\,\gamma + \quad \alpha\beta\gamma &= 0, \\ \overline{31}^2\,.\,\alpha + \overline{23}^2\,.\,\beta + \ 0\,.\,\gamma + \quad \alpha\beta\gamma &= 0, \\ \alpha + \quad \beta + \quad \gamma + 0\,.\,\alpha\beta\gamma &= 0; \end{aligned}$$

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More information

from which, eliminating α, β, γ, $\alpha\beta\gamma$ by the general theory of simple equations,

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The (additional) equation that exists between the distances of five points on a sphere or four points in a circle, has such a remarkable analogy with the preceding, that they almost require to be noticed at the same time.

If α, β, γ, r be the coordinates of the centre, and the radius of the sphere, and $\delta = \alpha^2 + \beta^2 + \gamma^2 - r^2$, we have immediately

$$\begin{array}{l} x_1^2 + y_1^2 + z_1^2 - 2\alpha x_1 - 2\beta y_1 - 2\gamma z_1 + \delta = 0, \\ \vdots \\ x_5^2 + y_5^2 + z_5^2 - 2\alpha x_5 - 2\beta y_5 - 2\gamma z_5 + \delta = 0; \end{array}$$

whence eliminating α, β, γ, δ,

$$\begin{vmatrix} x_1^2 + y_1^2 + z_1^2, & -2x_1, & -2y_1, & -2z_1, & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ x_5^2 + y_5^2 + z_5^2, & -2x_5, & -2y_5, & -2z_5, & 1 \end{vmatrix} = 0;$$

whence, multiplying by

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & x_1^2 + y_1^2 + z_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1, & x_5, & y_5, & z_5, & x_5^2 + y_5^2 + z_5^2 \end{vmatrix}$$

we have immediately

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0.$$

Forming the analogous equation for four points in a circle, and expanding, we readily deduce

$$\overline{14}^4\,\overline{23}^4 + \overline{12}^4\,\overline{34}^4 + \overline{13}^4\,\overline{24}^4 - 2\,.\,\overline{12}^2\,\overline{34}^2\,\overline{13}^2\,\overline{24}^2 - 2\,.\,\overline{14}^2\,\overline{23}^2\,\overline{13}^2\,\overline{24}^2 - 2\,.\,\overline{14}^2\,\overline{23}^2\,\overline{12}^2\,\overline{34}^2 = 0,$$

which is the rational, and therefore analytically the most simple form of

$$\overline{12}\;\overline{34} + \overline{14}\;\overline{23} = \overline{13}\;\overline{24}.$$

Euclid, B. vi., last proposition.

(It may be remarked that the two factors we have employed in the preceding eliminations, only differ by a numerical factor.)

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2.

ON THE PROPERTIES OF A CERTAIN SYMBOLICAL EXPRESSION.

[From the *Cambridge Mathematical Journal*, vol. III. (1841), pp. 62—71.]

THE series

$$\mathop{S}_{p}{}_0^\infty . \zeta_p (a^2+b^2 \ldots \mathfrak{n} \text{ terms})^{p+i} \left(\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \ldots\right)^p \frac{1}{\{(1+l)\,a^2+(1+m)\,b^2 \ldots\}^i}$$

$$\left(\zeta_p = \frac{1}{2^{2p+1}\, 1 \,.\, 2 \ldots p \,.\, i\,(i+1) \ldots (i+p)}\right) \ldots\ldots (\psi),$$

possesses some remarkable properties, which it is the object of the present paper to investigate. We shall prove that the symbolical expression (ψ) is independent of a, b, &c., and equivalent to the definite integral

$$\int_0^1 \frac{x^{2i-1}\,dx}{\{(1+lx^2)(1+mx^2) \ldots\}^{\frac{1}{2}}},$$

a property which we shall afterwards apply to the investigation of the attractions of an ellipsoid upon an external point, and to some other analogous integrals. The demonstration of this, which is one of considerable complexity, may be effected as follows:

Writing the symbol $\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \ldots$ under the form

$$\left(\frac{d^2}{da^2}+\frac{d^2}{db^2} \ldots\right) - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \ldots\right) = \Delta - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \ldots\right) \text{ suppose,}$$

let the p^{th} power of this quantity be expanded in powers of Δ. The general term is

$$(-1)^q . \frac{p\,(p-1) \ldots (p-q+1)}{1 \,.\, 2 \ldots q} . \Delta^{p-q} \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \ldots\right)^q,$$

which is to be applied to $\dfrac{1}{\{(1+l)\,a^2 \ldots\}^i}$.

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Considering the expression

$$\left(\frac{1}{1+l}\cdot\frac{d^2}{da^2}\cdots\right)^q\frac{1}{\{(1+l)\,a^2\ldots\}^i};$$

if for a moment we write

$$(1+l)\,a^2=a_1^2,\ \text{\&c.};\quad \Delta_1=\frac{d^2}{da_1^2}+\frac{d^2}{db_1^2}\cdots;\quad \rho_1=a_1^2+b_1^2\ldots,$$

this becomes $$\Delta_1^q\frac{1}{\rho_1^i}.$$

Now it is immediately seen that $\Delta_1\dfrac{1}{\rho_1^{i'}}=\dfrac{2i'(2i'+2-\mathfrak{n})}{\rho_1^{i+1}}$;

from which we may deduce

$$\Delta_1^q\frac{1}{\rho_1^i}=\frac{2i(2i+2)\ldots(2i+2q-2)(2i+2-\mathfrak{n})\ldots(2i+2q-\mathfrak{n})}{\rho_1^{i+q}},$$

or, restoring the value of ρ_1, and forming the expression for the general term of (ψ), this is

$$\zeta_p\,.\,\rho^{p+1}\left\{\begin{array}{l}\qquad\qquad\Delta^p\dfrac{1}{(a^2+b^2\ldots+la^2+mb^2+\text{\&c.})^i}\\[2ex] -\dfrac{p}{1}2i(2i+2-\mathfrak{n})\,\Delta^{p-1}\dfrac{1}{(a^2+b^2+\ldots+la^2+mb^2+\ldots)^i}\\[2ex] +\text{\&c.}\end{array}\right.$$

ρ representing the quantity $a^2+b^2+\text{\&c.}$

Hence, selecting the terms of the s^{th} order in l, m, &c. the expression for the part of (ψ) which is of the s^{th} order in l, m &c. may be written under the form

$$\mathop{S}\nolimits_{p\,0}^{\;s}\frac{(-1)^s\rho^{p+1}\zeta_p}{1\,.\,2\ldots s}$$

multiplied by

$$\left\{\begin{array}{l}\qquad\qquad i(i+1)\ldots(i+s-1)\,\Delta^p\dfrac{U}{\rho^{i+s}}\\[2ex] \qquad -\dfrac{p}{1}2i(2i+2-\mathfrak{n})(i+1)\ldots\quad(i+s)\,\Delta^{p-1}\dfrac{U}{\rho^{i+s+1}}\\[2ex] +\dfrac{p(p-1)}{1\,.\,2}2i(2i+2)(2i+2-\mathfrak{n})(2i+4-\mathfrak{n})(i+2)\ldots(i+s+1)\,\Delta^{p-2}\dfrac{U}{\rho^{i+s+2}}\\[2ex] -\text{\&c.}\qquad [la^2+mb^2\ldots=U\text{ suppose}]\end{array}\right.$$

which for conciseness we shall represent by

$$\frac{(-1)^s}{1\,.\,2\ldots s}\mathop{S}\nolimits_{p\,0}^{\;s}\rho^{p+1}\,.\,\zeta_p\left\{\begin{array}{l}\qquad\quad\alpha_s\,\Delta^p\dfrac{U}{\rho^{i+s}}\\[2ex] \quad-\dfrac{p}{1}\beta_s\,\Delta^{p-1}\dfrac{U}{\rho^{i+s+1}}\\[2ex] +\dfrac{p(p-1)}{1\,.\,2}\gamma_s\,\Delta^{p-2}\dfrac{U}{\rho^{i+s+2}}\\[2ex] -\text{\&c.}\end{array}\right.$$

$=S$ suppose.

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Excerpt
More information

Now U representing any homogeneous function of the order $2s$, it is easily seen that

$$\Delta\frac{U}{\rho^i}=\frac{\Delta U}{\rho^i}+2i\,(2i+2-4s-\mathfrak{n})\,\frac{U}{\rho^{i+1}};$$

and repeating continually the operation Δ, observing that ΔU, $\Delta^2 U$, &c. are of the orders $2(s-1)$, $2(s-2)$, &c. we at length arrive at

$$\begin{aligned}\Delta^q.\frac{U}{\rho^i}=\Delta^q\,U.\frac{1}{\rho^i}&\\ &+\frac{q}{1}\,2i\,(2i+2q-4s-\mathfrak{n})\,\Delta^{q-1}\,U.\frac{1}{\rho^{i+1}}\\ &+\frac{q\,(q-1)}{1.2}\,2i\,(2i+2)\,(2i+2q-4s-\mathfrak{n})\,(2i+2q-4s-\mathfrak{n}-2)\,\Delta^{q-2}\,U.\frac{1}{\rho^{i+2}}\\ &\;\vdots\\ &+2i\,(2i+2)\dots(2i+2q)\,(2i+2q-4s-\mathfrak{n})\dots\quad(2i+2-4s-\mathfrak{n})\,U.\frac{1}{\rho^{i+q}}.\end{aligned}$$

Changing i into $s+i+i'$, we have an equation which we may represent by

$$\Delta^q\,\frac{U}{\rho^{s+i+i'}}=A_{q,\,i'}\,\frac{\Delta^q\,U}{\rho^{s+i+i'}}+{}^1A_{q,\,i'}\,\frac{\Delta^{q-1}\,U}{\rho^{s+i+i'+1}}\dots+{}^qA_{q,\,i'}\,\frac{U}{\rho^{s+i+i'+q}}\dots\qquad(\alpha),$$

where in general

$$\begin{aligned}{}^rA_{q,\,i'}=\;&\frac{q\,(q-1)\dots(q-r+1)}{1.2\dots r}\\ &\times(2s+2i+2i')\,(2s+2i'+2i+2)\dots(2s+2i'+2i+2r-2)\\ &\times(2i+2i'+2q-2s-\mathfrak{n})\dots(2i+2i'+2q-2s-\mathfrak{n}-2r+2).\end{aligned}$$

Now the value of S, written at full length, is

$$\frac{(-1)^s}{1.2\dots s}\left\{\begin{aligned}&\zeta_s\rho^{s+1}\left(\alpha_s\Delta^s\frac{U}{\rho^{s+1}}-\frac{s}{1}\beta_s\Delta^{s-1}\,\frac{U}{\rho^{s+i+1}}\dots\right.\\ &+\zeta_{s-1}\,\rho^{s+i-1}\left(\alpha_s\Delta^{s-1}\frac{U}{\rho^{s+1}}-\frac{s-1}{1}\,\beta_s\Delta^{s-2}\,\frac{U}{\rho^{s+i+1}}+\dots\right.\\ &+\text{\&c.}\end{aligned}\right.$$

and substituting for the several terms of this expansion the values given by the equation (α), we have

$$S=\frac{(-1)^s}{1.2\dots\dots s}\left(k_0\Delta^s U+k_1\frac{1}{\rho}\Delta^{s-1}U\dots\dots+k_s\frac{1}{\rho^s}\,U\right)$$

where in general

$$\begin{aligned}k_x=\;&\alpha_s\,({}^xA_{s,\,0}\,\zeta_s+{}^{x-1}A_{s-1,\,0}\,\zeta_{s-1}\dots+A_{s-x,\,0}\,\zeta_{s-x})\\ &-\beta_s\left(\frac{s}{1}\,{}^{x-1}A_{s-1,\,1}\,\zeta_s\dots+\frac{(s-x+1)}{1}\,A_{s-x,\,1}\,\zeta_{s-x+1}\right)\\ &\;\vdots\\ &\pm\lambda_s\left(\frac{s\,(s-1)\dots(s-x+1)}{1.2\dots x}\,A_{s-x,\,x}\,\zeta_s\right),\end{aligned}$$

λ_s being the $(x+1)^{\text{th}}$ of the series $\alpha_s, \beta_s\dots$

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Excerpt
More information

Substituting for the quantities involved in this expression, and putting, for simplicity $2i+2-\mathfrak{n}=2\gamma$, we have, without any further reduction, except that of arranging the factors of the different terms, and cancelling those which appear in the numerator and denominator of the same term,

$$\frac{(-1)^s k_x}{1.2\ldots\ldots s} = \frac{(-1)^{s-x}(1-\gamma)(2-\gamma)\ldots(x-\gamma)}{2^{2s+1}.1.2\ldots s.1.2\ldots(s-x).1.2\ldots x}$$

multiplied by the series

$$(i+s+1)\ldots(i+s+x-1) \text{ into}$$

$$\left\{1+\frac{\gamma}{1}\frac{x}{x-\gamma}+\frac{\gamma(\gamma+1)}{1.2}\frac{x(x-1)}{(x-\gamma)(x-1-\gamma)}+\ldots \qquad (x+1) \text{ terms}\right\}$$

$$-\frac{(i+s)\ldots(i+s+x-2)}{1-\gamma} \text{ into}$$

$$\left\{x+\frac{\gamma}{1}\frac{x(x-1)}{x-\gamma}+\frac{\gamma(\gamma+1)}{1.2}\frac{x(x-1)(x-2)}{(x-\gamma)(x-1-\gamma)}+\ldots \qquad x \text{ terms}\right\}$$

$$\vdots$$

$$+(-1)^r\frac{(i+s-r+1)\ldots(i+s+x-r-1)}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)} \text{ into}$$

$$\left\{x(x-1)\ldots(x-r+1)+\frac{\gamma}{1}\frac{x(x-1)\ldots(x-r)}{x-\gamma}+\ldots(x+r-1) \text{ terms}\right\}$$

to $r=x$.

Now it may be shown that

$$\frac{1}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)}$$

$$\left\{x(x-1)\ldots(x-r+1)+\frac{\gamma}{1}\frac{x(x-1)\ldots(x-r)}{x-\gamma}+\&c.\ldots(x+1-r) \text{ terms}\right\}$$

$$=\frac{x(x-1)\ldots(r+1).x(x-1)\ldots(x-r+1)}{(1-\gamma)(2-\gamma)\ldots(x-\gamma)},$$

which reduces the expression for k_x to the form

$$\frac{(-1)^s k_x}{1.2\ldots s} = \frac{(-1)^{s+x}}{2^{2s+1}.1.2\ldots s.1.2\ldots(s-x)}\left\{\begin{array}{l}(i+s+1)\ldots(i+s+x-1)\\ -\frac{x}{1}(i+s)\ldots(i+s+x-2)\\ +\frac{x(x-1)}{1.2}(i+s-1)\ldots(i+s+x-3)\\ \pm\&c.\ (x+1) \text{ terms};\end{array}\right.$$

from which it may be shown, that except for $x=0$, $k_x=0$.

The value $x=0$, observing that the expression

$$(i+s+1)(i+s+2)\ldots(i+s-1)$$

represents $\frac{1}{i+s}$, gives

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$$\frac{(-1)^s k_0}{1.2\ldots s} = \frac{(-1)^s}{2^{2s}(1.2\ldots s)^2.(2i+2s)};$$

or we have simply

$$S = \frac{(-1)^s}{2^{2s}(1.2\ldots s)^2.(2i+2s)}\Delta^s U,$$

where $$\Delta = \frac{d^2}{da^2} + \frac{d^2}{db^2} + \ldots,\quad U = (la^2 + mb^2 \ldots)^s.$$

Consider the term $$\frac{1.2\ldots s}{1.2\ldots\lambda.1.2\ldots\mu.\&c.}a^{2\lambda}b^{2\mu}\ldots l^\lambda m^\mu\ldots;$$

with respect to this, Δ^s reduces itself to

$$\frac{1.2\ldots s}{1.2\ldots\lambda.1.2\ldots\mu.\&c.}\left(\frac{d}{da}\right)^{2\lambda}\ldots$$

and the corresponding term of S is

$$\frac{(-1)^s}{2^{2s}(2i+2s)(1.2\ldots\lambda.1.2\ldots\mu.\ \&c.)^2}1.2\ldots 2\lambda.1.2\ldots 2\mu.\ \&c.\ l^\lambda m^\mu\ldots$$

$$= \frac{(-1)^s.1.3\ldots(2\lambda-1).1.3\ldots(2\mu-1).\ \&c.}{(2i+2s)\ 2.4\ldots 2\lambda.2.4\ldots 2\mu.\ \&c.}l^\lambda m^\mu\ldots$$

which, omitting the factor $\frac{1}{2i+2s}$, and multiplying by x^{2s}, is the general term of the s^{th} order in $l, m, \ldots$ of

$$\frac{1}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}}.$$

The term itself is therefore the general term of

$$\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}};$$

or taking the sum of all such terms for the complete value of S, and the sum of the different values of S for the values 0, 1, 2… of the variable s, we have the required equation

$$\psi = \int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}}.$$

Another and perhaps more remarkable form of this equation may be deduced by writing $\frac{a^2}{1+l}$, $\frac{b^2}{1+m}$, &c. for a^2, b^2, &c., and putting $\frac{a^2}{1+l} + \frac{b^2}{1+m} + \&c. = \eta^2$, $l\eta^2 = \alpha^2$, $m\eta^2 = \beta^2$, &c.: we readily deduce

$$\eta^{n-2i}\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(\eta^2+\alpha^2x^2)(\eta^2+\beta^2x^2)\ldots\}}}$$

$$= \mathop{S}_p{}_0^\infty \frac{1}{2^{2p+1}.1.2\ldots p.i(i+1)\ldots(i+p)}\left(\alpha^2\frac{d^2}{da^2} + \beta^2\frac{d^2}{db^2}\ldots\right)^p \frac{1}{(a^2+b^2\ldots)^i},$$

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η being determined by the equation

$$\frac{a^2}{\eta^2+\alpha^2}+\frac{b^2}{\eta^2+\beta^2}\ldots=1;$$

or, as it may otherwise be written,

$$\eta^2=a^2+b^2+\ldots-\frac{a^2\alpha^2}{\eta^2+\alpha^2}-\frac{b^2\beta^2}{\eta^2+\beta^2}-\&c.$$

$\mathfrak{n}$, it will be recollected, denotes the number of the quantities a, b, &c.

Now suppose

$$V=\iint\ldots\phi(a-x,\ b-y,\ldots)\,dx\,dy\ldots$$

(the integral sign being repeated $\mathfrak{n}$ times) where the limits of the integral are given by the equation

$$\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\&c.=1;$$

and that it is permitted, throughout the integral to expand the function $\phi(a-x,\ldots)$ in ascending powers of x, y, &c. (the condition for which is apparently that of ϕ not becoming infinite for any values of x, y, &c., included within the limits of the integration): then observing that any integral of the form $\iint\ldots x^p y^q\ldots dx\,dy$ &c.... where any one of the exponents p, q, &c.... is odd, when taken between the required limits contains equal positive and negative elements and therefore vanishes, the general term of V assumes the form

$$\frac{1}{1.2\ldots 2r.1.2\ldots 2s\ldots}\left(\frac{d}{da}\right)^{2r}\left(\frac{d}{db}\right)^{2s}\ldots\phi(a,\ b\ldots)\iint\ldots x^{2r}y^{2s}\ldots dx\,dy\ldots$$

Also, by a formula quoted in the eleventh No. of the *Mathematical Journal*, the value of the definite integral $\iint\ldots x^{2r}y^{2s}\ldots dx\,dy\ldots$ is

$$h^{2r+1}h_{,}^{2s+1}\ldots\frac{\Gamma(r+\frac{1}{2})\,\Gamma(s+\frac{1}{2})\ldots}{\Gamma(r+s+\ldots+\frac{1}{2}\mathfrak{n}+1)},$$

(observing that the value there given referring to positive values only of the variables, must be multiplied by $2^{\mathfrak{n}}$): or, as it may be written

$$h^{2r+1}h_{,}^{2s+1}\ldots\pi^{\frac{1}{2}\mathfrak{n}}.\frac{1}{2^{r+s\ldots}}\frac{1.3\ldots(2r-1).1.3\ldots(2s-1)\ldots}{\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\ldots(\frac{1}{2}\mathfrak{n}+r+s\ldots)\,\Gamma(\frac{1}{2}\mathfrak{n})}:$$

hence the general term of V takes the form

$$\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\,\frac{1}{\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\ldots(\frac{1}{2}\mathfrak{n}+r+s\ldots)}.\frac{1}{2^{2r+2s\ldots}}\,\frac{1}{1.2.3\ldots r.1.2\ldots s\ldots}$$

$$\times\left(h^2\frac{d^2}{da^2}\right)^r\left(h_{,}^2\frac{d^2}{db^2}\right)^s\ldots\phi(a,\ b,\ldots);$$

and putting $r+s+\&c.=p$, and taking the sum of the terms that answer to the same value of p, it is immediately seen that this sum is

$$=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}.\frac{1}{2^{2p}.1.2\ldots p.\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\ldots(\frac{1}{2}\mathfrak{n}+p)}\left(h^2\frac{d^2}{da^2}+h_{,}^2\frac{d^2}{db^2}\ldots\right)^p\phi(a,\ b\ldots).$$

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