STATICS

Chapter I

INTRODUCTION

1·1. Statics is that branch of Mechanics which is concerned with the conditions under which bodies remain at rest relative to their surroundings. In such circumstances bodies are said to be in a state of equilibrium. It is assumed that they are acted upon by 'forces' which balance one another. The primary conceptions of Statics are forces and the bodies upon which they act. In Dynamics, force is defined as that which changes or tends to change the state of motion of a body, but in Statics we are not concerned with motion save in so far as a force would cause motion if unbalanced by another force. We get our ideas of force from the ability which we ourselves possess to cause, or to resist, the motion of our own bodies or of other bodies. We are conscious of measurable efforts required to lift bodies or to support bodies which would otherwise fall to the ground; we speak of these measurable efforts as forces which we exert, and we compare them with the 'weights' that we associate with bodies, by which term we mean 'the forces with which the Earth attracts them'. Thus the phrase 'a force of x pounds weight' means a force that would support a body weighing x pounds.

When holding the string of a kite flying in a gusty wind we are conscious of a pull or 'tension' on the string at the point where it leaves the hand that holds it, and we realize that this 'force' is something measurable and of varying measure, and that it acts now in this direction and now in that as the string moves hither and thither with the kite. This serves to illustrate the fact that a force possesses magnitude and acts in a definite line (that of the string) and may be regarded as acting at a definite point of that line (the point where the string leaves the hand). Thus in so far as a force possesses magnitude and direction it is a **vector** as defined in the next chapter, but whether forces obey the vector law of addition must be discussed later.

1·2. It is necessary to specify at the outset what it is upon which forces act. They cannot act upon nothing. Forces act upon material bodies, and unless the contrary is stated we shall assume that in every case the bodies are *rigid*, i.e. that the distance between each pair of particles that compose a body remains invariable. Actual bodies are all more or less elastic and capable of compression, extension or distortion under the action of forces, and the assumption of perfect rigidity is necessary in order to simplify the building up of the elementary theory of the subject.

We regard solid bodies as agglomerations of particles held together by forces of cohesion. We do not put any limit to the number, large or small, of particles that go to form a body, and there is no difficulty in the conception of a force acting upon a minute body or 'a single particle'.

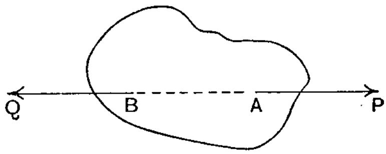
1·3. The forces with which we shall be concerned are of three types: (*a*) a *push* or *thrust*; (*b*) a *pull* or *tension*; (*c*) an *attraction* such as the 'weight' of a body or 'the force with which the Earth attracts it'.

1·4. We shall assume (i) that if two equal and opposite forces act upon a body in the same straight line they have no effect upon the body's state of rest or motion, i.e. they balance one another. This is a statement the truth of which can easily be tested by experiment.

We shall also assume (ii) that the forces mutually exerted between two bodies are always equal and opposite. This is the *law of reaction* enunciated by Newton in the words 'Action and Reaction are equal and opposite'. It means that if a body A exerts a force F upon a body B, then B exerts an equal force F upon A but in the opposite direction.

1·5. A consequence of assumption (i) of **1·4** is what is sometimes called the *Principle of Transmissibility of Force*, viz. that a force may be supposed to act at *any* point in its line of action provided that the point is rigidly connected with the body on which the force acts.

For if P and Q are equal and opposite forces acting upon a body at the points A and B, then we may say that Q balances P no matter at what point B in the line AB the force Q acts; i.e. a force Q acting at A produces the same effect as a force Q acting at B in the same straight line.



1·6. In the following chapter we shall discuss some of the common properties of a class of physical magnitudes called *vectors* and in Chapter III we shall give further reasons for including *force* in this class and then develop the consequences.

Chapter II

VECTORS

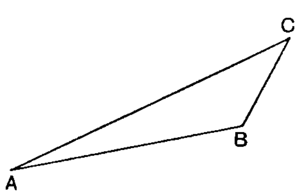
2·1. The physical quantities or measurable objects of reasoning in Applied Mathematics are of two classes. The one class, called **Vectors**, consists of all measurable objects of reasoning which possess directional properties, such as *displacement*, *velocity*, *acceleration*, *momentum*, *force*, etc. The other class, called Scalars, comprises measurable objects of reasoning which possess no directional properties, such as *mass*, *work*, *energy*, *temperature*, etc.

The simplest conception of a vector is associated with the displacement of a point. Thus the displacement of a point from A to B may be represented by the line AB, where the length, direction and sense (AB not BA) are all taken into account. Such a displacement is called a *vector* (Latin *veho*, I carry). A vector may be denoted by a single letter, e.g. as when we speak of 'the force P', or 'the acceleration f', or by naming the line, such as AB, which represents the vector. When it is desired to indicate that symbols denote vectors it is usual to *print* them in Clarendon type, e.g. **P**, and to *write* them with a bar above the symbol, e.g. $\overline{P}$, $\overline{AB}$.

Since the displacement from B to A is the opposite of a displacement from A to B, we write

$$\overline{BA} = -\overline{AB},$$

and take vectors in opposite senses to have opposite signs. Since two successive displacements of a point from A to B and from B to C produce the same result as a single displacement from A to C, we say that the vector AC is equal to the sum of the vectors AB, BC and write



$$\overline{AC} = \overline{AB} + \overline{BC} \quad \text{......(1)},$$

and further, if A, B, C, ... K, L are any set of points,

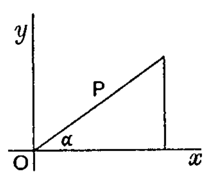
$$\overline{AL} = \overline{AB} + \overline{BC} + \ldots + \overline{KL} \quad \text{...............(2)}.$$

Vectors in general are not localized; thus we may have a displacement of an assigned length in an assigned direction and sense but its locality not specified. In such a case all equal and parallel lines in the same sense will represent the same vector. On the other hand, vectors may be localized, either at a point, e.g. the *velocity* of a particle; or in a line, as for example a *force* whose line of action (but not point of application) is specified.

2·2. Composition of Vectors. A single vector which is equivalent to two or more vectors is called their **resultant**, and they are called the **components** of the resultant. Vectors are compounded by geometrical addition as indicated in (1) and (2) of the preceding Article.

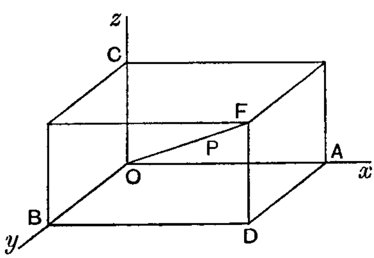
A vector can be resolved into two components in assigned directions in any plane which contains the vector; for if AC be the vector, and through A, C two lines are drawn in the assigned directions meeting in B, then AB, BC are the components required.

When a vector is resolved into two components in directions at right angles to one another, each component is called the **resolved part of the vector** in the direction specified. Thus if a vector **P** makes an angle α with a given direction Ox, the resolved parts of **P** in the direction Ox and in the perpendicular direction Oy are



$$\mathbf{P}\cos\alpha \quad \text{and} \quad \mathbf{P}\sin\alpha.$$

Further, if Ox, Oy, Oz are three lines mutually at right angles, and the line OF represents a vector **P**, if we construct a rectangular parallelepiped with OF as diagonal and edges OA, OB, OC along Ox, Oy, Oz, as in the figure, then



$$\overline{OF} = \overline{OA} + \overline{AD} + \overline{DF}$$

or $$\mathbf{P} = \overline{OA} + \overline{OB} + \overline{OC}.$$

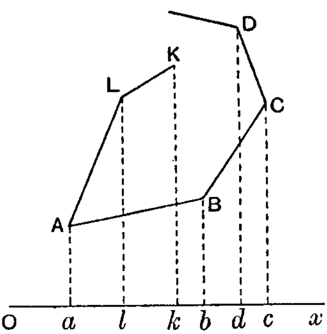
And, if **P** makes angles α, β, γ with Ox, Oy, Oz, we have, since OAF is a right angle, $\overline{OA} = \mathbf{P}\cos\alpha$, and similarly $\overline{OB} = \mathbf{P}\cos\beta$

and $\overline{OC} = \mathbf{P}\cos\gamma$, so that in this three-dimensional resolution of a vector its resolved parts in the three mutually perpendicular directions are

$$\mathbf{P}\cos\alpha,\quad \mathbf{P}\cos\beta,\quad \mathbf{P}\cos\gamma.$$

It is clear therefore that the resolved part of a vector along a given line is the orthogonal projection of the vector upon that line.

2·3. Let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, ... $\overline{KL}$ be a set of vectors forming sides of a polygon. Their resultant is the vector $\overline{AL}$ which completes the polygon. Let $a, b, c, d, \ldots k, l$ be the orthogonal projections of the points $A, B, C, D, \ldots K, L$ on any straight line Ox. Then, with due regard to signs,



$$ab + bc + cd + \ldots + kl = al.$$

But these projections ab, bc, ... are the resolved parts of the vectors $\overline{AB}$, $\overline{BC}$, ... in the direction Ox, therefore the algebraical sum of the resolved parts of a set of vectors in an assigned direction is equal to the resolved part of their resultant in the same direction.

2·4. Analytical Method. To compound n vectors

$$\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_n.$$

(i) When the vectors are in the same plane.

Let the vectors make angles $\alpha_1, \alpha_2, \ldots \alpha_n$ with an axis Ox. Each vector may be resolved into two components, one in the direction Ox and the other in the perpendicular direction Oy. The components in direction Ox are equivalent to a single vector

$$\mathbf{X} = \mathbf{P}_1\cos\alpha_1 + \mathbf{P}_2\cos\alpha_2 + \ldots + \mathbf{P}_n\cos\alpha_n = \Sigma(\mathbf{P}\cos\alpha),$$

and the components in direction Oy are equivalent to a single vector

$$\mathbf{Y} = \mathbf{P}_1\sin\alpha_1 + \mathbf{P}_2\sin\alpha_2 + \ldots + \mathbf{P}_n\sin\alpha_n = \Sigma(\mathbf{P}\sin\alpha).$$

The two vectors **X**, **Y** can now be compounded into a single vector **R** making an angle θ with Ox, such that

$$\mathbf{R}\cos\theta = \mathbf{X} \quad\text{and}\quad \mathbf{R}\sin\theta = \mathbf{Y},$$

and therefore

$$R^2 = X^2 + Y^2 \quad\text{and}\quad \tan\theta = Y/X \quad \text{.........(1)}.$$

(ii) When the vectors are not all in the same plane.

As in **2·2** take three axes Ox, Oy, Oz mutually at right angles and let the vectors make angles $\alpha_1, \alpha_2, \ldots \alpha_n$ with Ox, $\beta_1, \beta_2, \ldots \beta_n$ with Oy and $\gamma_1, \gamma_2, \ldots \gamma_n$ with Oz. Each vector may then be resolved into components of the types

$$\mathbf{P}\cos\alpha, \quad \mathbf{P}\cos\beta, \quad \mathbf{P}\cos\gamma$$

in the directions Ox, Oy, Oz. The components in direction Ox are equivalent to a single vector

$$\mathbf{X} = \mathbf{P}_1\cos\alpha_1 + \mathbf{P}_2\cos\alpha_2 + \ldots + \mathbf{P}_n\cos\alpha_n = \Sigma(\mathbf{P}\cos\alpha),$$

similarly the components in directions Oy and Oz are equivalent to single vectors

$$\mathbf{Y} = \mathbf{P}_1\cos\beta_1 + \mathbf{P}_2\cos\beta_2 + \ldots + \mathbf{P}_n\cos\beta_n = \Sigma(\mathbf{P}\cos\beta)$$

and

$$\mathbf{Z} = \mathbf{P}_1\cos\gamma_1 + \mathbf{P}_2\cos\gamma_2 + \ldots + \mathbf{P}_n\cos\gamma_n = \Sigma(\mathbf{P}\cos\gamma).$$

The three vectors **X**, **Y**, **Z** can now be compounded into a single vector **R** making angles θ, ϕ, ψ with Ox, Oy, Oz, such that

$$\mathbf{R}\cos\theta = \mathbf{X}, \quad \mathbf{R}\cos\phi = \mathbf{Y} \quad\text{and}\quad \mathbf{R}\cos\psi = \mathbf{Z} \quad \text{...(2)},$$

and by squaring and adding

$$R^2 = X^2 + Y^2 + Z^2 \quad \text{..................(3)}.$$

When the magnitude of R has been found from (3) its direction is determined by (2).

In obtaining (3) we have assumed that $\cos^2\theta + \cos^2\phi + \cos^2\psi = 1$; that this is true is seen from the figure of **2·2**, where θ, ϕ, ψ may denote the inclinations of OF to the axes, then

$$\cos^2\theta + \cos^2\phi + \cos^2\psi = \frac{OA^2}{OF^2} + \frac{OB^2}{OF^2} + \frac{OC^2}{OF^2} = 1.$$

2·41. The method of obtaining the resultant in **2·4** is based on the fact that if the vectors are all resolved in any assigned direction Ox, then the resolved part of the resultant in that direction is equal to the algebraical sum of the resolved parts of the given vectors.

When the vectors are not all in the same plane each vector is resolved into three components in the directions of three rectangular axes

Ox, Oy, Oz chosen arbitrarily, so that for any direction Ox in space the resolved part of the resultant is equal to the algebraical sum of the resolved parts of the given vectors. Also, if each vector be resolved into *two* components only, one along Ox and the other in the perpendicular plane yOz, the latter components taken together are equivalent to the resolved part of the resultant in the plane yOz.

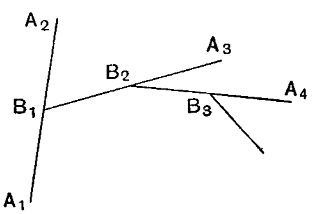
2·5. Vectors may be multiplied and divided by scalar numbers. Thus, if we take n equal vectors $\overline{AB}$ and compound them together, we get a vector $\overline{AC}$ such that $\overline{AC}=n\overline{AB}$; and conversely $\overline{AB}=\frac{1}{n}\overline{AC}$.

Note that relations of the form

$$\overline{AC}=n\overline{AB}, \quad \text{or} \quad p\overline{AB}+q\overline{AC}=0,$$

imply that the points A, B, C are in the same straight line.

2·6. Centroids or Mean Centres. If $m_1, m_2, m_3, \ldots m_n$ be a set of scalar magnitudes associated with a set of points $A_1, A_2, A_3, \ldots A_n$, the centroid or mean centre of the points for the given magnitudes is the point obtained by the following process: Divide A_1A_2 at B_1 so that



$$m_1A_1B_1=m_2B_1A_2;$$

divide B_1A_3 at B_2 so that

$$(m_1+m_2)B_1B_2=m_3B_2A_3;$$

divide B_2A_4 at B_3 so that $(m_1+m_2+m_3)B_2B_3=m_4B_3A_4$. Proceed in this way until all the points have been connected, then the last point of division B_{n-1}, usually denoted by the letter G, is called the *centroid* or *mean centre*.

2·61. In order to shew that this process leads in general to a unique point, i.e. that the point determined by the process is independent of the order in which the points $A_1, A_2, \ldots A_n$ are joined, we shall first prove that

$$m_1\overline{A_1G}+m_2\overline{A_2G}+\ldots+m_n\overline{A_nG}=0 \quad \ldots\ldots(1).$$

Assume that this formula is true for the first r points, i.e. that

$$m_1\overline{A_1B_{r-1}}+m_2\overline{A_2B_{r-1}}+\ldots+m_r\overline{A_rB_{r-1}}=0.$$

Now the next step in the process is to divide $B_{r-1}A_{r+1}$ at B_r so that

$$(m_1+m_2+\dots+m_r)\,\overline{B_{r-1}B_r}=m_{r+1}\overline{B_rA_{r+1}},$$

therefore, by adding the last two lines,

$$m_1\overline{A_1B_r}+m_2\overline{A_2B_r}+\dots+m_r\overline{A_rB_r}+m_{r+1}\overline{A_{r+1}B_r}=0.$$

It follows that if the formula (1) is true for the centroid of r points it is also true for the centroid of $r+1$ points; but it is true for two points, since, by hypothesis,

$$m_1\overline{A_1B_1}+m_2\overline{A_2B_1}=0.$$

Therefore the formula (1) is true for the centroid of any number of points.

Now, if by taking the points in a different order we arrive at a centroid G', we can shew similarly that

$$m_1\overline{A_1G'}+m_2\overline{A_2G'}+\dots+m_n\overline{A_nG'}=0 \quad \dots\dots(2);$$

and by subtracting (1) from (2) we get

$$(m_1+m_2+\dots+m_n)\,\overline{GG'}=0.$$

Hence G' must coincide with G unless $m_1+m_2+\dots+m_n=0$. In the latter case there is no centroid at a finite distance, because the last step in the process of finding the centroid consists in dividing a line in the ratio $m_n:m_1+m_2+\dots+m_{n-1}$, i.e. in the ratio $1:-1$.

2·7. Centroid Method of Compounding Vectors. To shew, with the notation of **2·6**, that, if O be any other point, the resultant of n vectors $m_1\overline{OA_1}$, $m_2\overline{OA_2}$, ... $m_n\overline{OA_n}$ is $(m_1+m_2+\dots+m_n)\,\overline{OG}$, where G is the centroid of the points $A_1, A_2, \dots A_n$ for the magnitudes $m_1, m_2, \dots m_n$.

This follows at once by substituting

$$\overline{OA_1}=\overline{OG}+\overline{GA_1},\quad \overline{OA_2}=\overline{OG}+\overline{GA_2},\ \text{etc.},$$

so that

$$m_1\overline{OA_1}+m_2\overline{OA_2}+\dots+m_n\overline{OA_n}$$
$$=(m_1+m_2+\dots+m_n)\,\overline{OG}+(m_1\overline{GA_1}+m_2\overline{GA_2}+\dots+m_n\overline{GA_n}),$$

and by **2·61** (1) the sum of the terms in the last bracket is zero, therefore

$$m_1\overline{OA_1}+m_2\overline{OA_2}+\dots+m_n\overline{OA_n}=(m_1+m_2+\dots+m_n)\,\overline{OG}.$$

2·71. When reference is made to the centroid of a set of points without mention of any associated magnitudes it is understood that the magnitudes are equal; thus the centroid of a triangle ABC is a point G such that

$$\overline{AG}+\overline{BG}+\overline{CG}=0.$$

2·72. It may be noticed that if **P**, **Q**, **R** are vectors in the lines OA, OB, OC, then the resultant vector is

$$\left(\frac{P}{OA}+\frac{Q}{OB}+\frac{R}{OC}\right)\overline{OG},$$

where G is the centroid of the points A, B, C for the magnitudes P/OA, Q/OB, R/OC; for a vector **P** in the line OA is the same as $\frac{P}{OA}\overline{OA}$.

2·8. We began this chapter with a statement that certain physical quantities were to be classed together as vectors and then proceeded to define the properties of vectors and shew how they can be compounded. In order, therefore, to satisfy ourselves that a physical quantity such as force or acceleration is rightly described as a vector, we need adequate reasons for stating that

(i) it possesses direction,

(ii) it conforms to the laws $\overline{AB}=-\overline{BA}$,

and $\overline{AB}+\overline{BC}=\overline{AC}$.

EXAMPLES

1. ABC is a triangle. Prove that the magnitude of the resultant of vectors $\overline{AB}$, $2\overline{BC}$ and $3\overline{CA}$ is $(b^2+c^2+2bc\cos A)^{\frac{1}{2}}$ and that its direction is that of the diagonal through A of the parallelogram of which AB, AC are adjacent sides.

2. $ABCDEF$ is a regular hexagon. Prove that

$$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=3\overline{AD}.$$

3. AA', BB', CC', DD' are parallel edges of a parallelepiped, of which AC' is a diagonal. Prove that

$$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AA'}+\overline{AB'}+\overline{AC'}+\overline{AD'}=4\overline{AC'}.$$

4. Prove that, if G is the middle point of AB and G' is the middle point of $A'B'$, then $\overline{AA'}+\overline{BB'}=2\overline{GG'}$.

5. Prove that, if G is the centroid of n points $A_1, A_2, \dots A_n$, and G' is the centroid of n points $B_1, B_2, \dots B_n$, then

$$\overline{A_1B_1}+\overline{A_2B_2}+\dots+\overline{A_nB_n}=n\overline{GG'}.$$