DYNAMICS

Chapter I

INTRODUCTION

1·1. The subject of Dynamics is generally divided into two branches: the first, called Kinematics, is concerned with the geometry of motion apart from all considerations of force, mass or energy; the second, called Kinetics, is concerned with the effects of forces on the motion of bodies.

1·2. In order to describe the motion of a body or of a point two things are needed, (i) a frame of reference, (ii) a time-keeper. It is not possible to describe absolute motion, but only motion relative to surrounding objects; and a suitable frame of reference depends on the kind of motion that it is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of axes with an origin at the Sun's centre.

1·3. It is important to realize that there is no such thing as absolute time, but the period of rotation of the Earth relative to the fixed stars provides a unit of time, *the sidereal day*, which, so far as it can be tested with other time measures, is constant and therefore adequate for the purposes of ordinary dynamics.

1·4. The functions involved in dynamical problems are for the most part differential coefficients with regard to 'time,' 't,' as the independent variable. Thus 'motion' is 'change of position' or 'displacement,' 'velocity' is 'rate of displacement' and 'acceleration' is 'rate of change of velocity.' Hence, if x denotes a distance, dx/dt denotes a velocity and d^2x/dt^2 denotes an acceleration. The formulation of a dynamical problem therefore in general consists of one or more relations between certain variables (coordinates of position) and their differential coefficients with regard to time. Such relations are called differential equations.

NOTE ON DIFFERENTIAL EQUATIONS

1·5. It is assumed that the reader is acquainted with the elementary processes of differentiation and integration.

A *differential equation* is a relation between an independent variable t, a dependent variable x, and one or more of the differential coefficients of x with regard to t. The *order* of a differential equation is that of the highest differential coefficient that it contains. A *solution* of a differential equation is a relation between x and t that satisfies the equation, and the *complete solution* of a differential equation is a relation between x, t and one or more arbitrary constants of integration, the number of such constants being equal to the order of the equation.

For example:

$$\text{(i)}\quad \frac{dx}{dt}-2x=0$$

is a differential equation of the first order. It will be found on substitution that $x=e^{2t}$ is a solution; and the complete solution is $x=Ce^{2t}$, where C is an arbitrary constant.

$$\text{(ii)}\quad \frac{d^2x}{dt^2}+x=0$$

is a differential equation of the second order. It has solutions

$$x=\sin t \quad\text{and}\quad x=\cos t,$$

and the complete solution is

$$x=A\sin t+B\cos t,$$

where A and B are arbitrary constants.

1·6. The differential equations of dynamics are of either the first or second order.

Equations of the First Order.

We may have to deal with equations in which the variables can be separated. Such equations can be put in the form

$$M\,dx/dt=N \quad\text{.................................(1)},$$

where M is a function of x only (or a constant) and N is a function of t only (or a constant). The complete solution is

$$\int M\,dx = \int N\,dt + C \quad \text{.........................(2)},$$

where C is an arbitrary constant.

For example, the equation

$$x\frac{dx}{dt} = g - kx^2$$

is solved by writing

$$\frac{x\,dx}{g-kx^2} = g\,dt,$$

so that

$$-\frac{1}{2k}\log(g-kx^2) = gt + C$$

is the complete solution.

1·61. Another type of equation that sometimes occurs in dynamics is the *linear equation of the first order*. A differential equation is said to be linear when it does not contain powers or products of the dependent variable x and its differential coefficients. Thus the linear equation of the first order is

$$dx/dt + Mx = N \quad \text{.............................(3)},$$

where M, N are functions of t or constants.

The solution is effected by first multiplying both sides of the equation by $e^{\int M dt}$ and then integrating; because it can easily be verified that

$$\frac{d}{dt}\left(xe^{\int M dt}\right) = \left(\frac{dx}{dt} + Mx\right)e^{\int M dt}$$

Hence $$xe^{\int M dt} = \int e^{\int M dt} N\,dt + C \text{.........................(4)},$$

where C is an arbitrary constant.

We note that if M is a constant the solution is

$$xe^{Mt} = \int e^{Mt} N\,dt + C \quad \text{.........................(5)}.$$

For example, the equation

$$\frac{dx}{dt} + kx = gt$$

can be integrated if both sides are multiplied by e^{kt}, giving on integration

$$xe^{kt} = g\int e^{kt}\,t\,dt + C$$
$$= ge^{kt}\left(\frac{t}{k} - \frac{1}{k^2}\right) + C,$$

or $$x = \frac{g}{k}\left(t - \frac{1}{k}\right) + Ce^{-kt} \text{.........................(6)}.$$

1·7. Equations of the Second Order.

A common type of differential equation of the second order is

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + bx = 0 \quad \text{.........................(7)},$$

where a and b are constants.

It is easily seen by substitution that $x=e^{mt}$ is a solution of this equation, provided that

$$m^2+2am+b=0 \quad\text{.............................(8)},$$

so that, if m_1, m_2 are the roots of this quadratic, the complete solution of (7) is

$$x=C_1e^{m_1t}+C_2e^{m_2t} \quad\text{.............................(9)},$$

where C_1, C_2 are two arbitrary constants.

The roots of (8) are $-a\pm\sqrt{(a^2-b)}$, and there are three cases to be considered:

(i) *Real roots.* $a^2-b=n^2$, say; then (9) may be written

$$x=e^{-at}(C_1e^{nt}+C_2e^{-nt}) \quad\text{.......................(10)}.$$

(ii) *Equal roots.* $a^2-b=0$. The form $e^{-at}(C_1+C_2)$ is inadequate for a *complete* solution, since C_1+C_2 can only be regarded as one arbitrary constant, and, since the differential equation is of the second order, the complete solution should contain two. It is easily verified however that, when $a^2=b$, the form

$$x=e^{-at}(C_1+C_2t) \quad\text{..........................(11)}$$

satisfies equation (7), and since it contains two arbitrary constants, it is the complete solution.

(iii) *Imaginary roots.* $a^2-b^2=-n^2$, say; (9) may now be written

$$x=e^{-at}(C_1e^{int}+C_2e^{-int}),$$

or $$x=e^{-at}\{(C_1+C_2)\cos nt+i(C_1-C_2)\sin nt\};$$

which again may be written in the more convenient form

$$x=e^{-at}(C\cos nt+C'\sin nt) \quad\text{....................(12)}.$$

Special cases of the foregoing. When $a=0$.

(α) The complete solution of

$$\frac{d^2x}{dt^2}-n^2x=0 \quad\text{.............................(13)}$$

is $$\left.\begin{aligned}x&=Ae^{nt}+Be^{-nt}\\ \text{or}\quad x&=C\cosh nt+D\sinh nt\end{aligned}\right\} \quad\text{.......................(14)},$$

where A, B or C, D are the arbitrary constants.

(β) The complete solution of

$$\frac{d^2x}{dt^2}+n^2x=0 \quad\text{.............................(15)}$$

is $$\left.\begin{aligned}x&=A\cos nt+B\sin nt\\ \text{or}\quad x&=C\cos(nt+\alpha)\\ \text{or}\quad x&=C'\sin(nt+\alpha')\end{aligned}\right\} \quad\text{.......................(16)},$$

where A, B or C, α or C', α' are the arbitrary constants

1·71. Numerical Examples.

(i) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0$; $x = Ae^t + Be^{3t}$.

(ii) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$; $x = e^{2t}(A + Bt)$.

(iii) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$; $x = e^{2t}(A\cos 3t + B\sin 3t)$.

(iv) $\frac{d^2x}{dt^2} - 4x = 0$; $x = Ae^{2t} + Be^{-2t}$.

(v) $\frac{d^2x}{dt^2} + 4x = 0$; $x = A\cos 2t + B\sin 2t$.

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Excerpt
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Chapter II

VECTORS

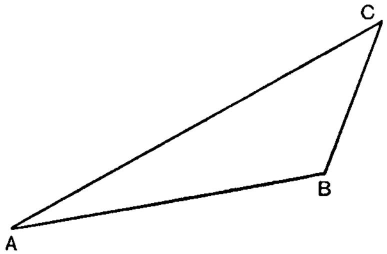
2·1. The physical quantities or measurable objects of reasoning in Applied Mathematics are of two classes. The one class, called **Vectors**, consists of all measurable objects of reasoning which possess directional properties, such as *displacement, velocity, acceleration, momentum, force,* etc. The other class, called **Scalars**, comprises measurable objects of reasoning which possess no directional properties, such as *mass, work, energy, temperature,* etc.

The simplest conception of a vector is associated with the displacement of a point. Thus the displacement of a point from A to B may be represented by the line AB, where the length, direction and sense (AB not BA) are all taken into account. Such a displacement is called a *vector* (Latin *veho*, I carry). A vector may be denoted by a single letter, e.g. as when we speak of 'the force P,' or 'the acceleration f,' or by naming the line, such as AB, which represents the vector. When it is desired to indicate that symbols denote vectors it is usual to *print* them in Clarendon type, e.g. **P**, and to *write* them with a bar above the symbol, e.g. $\overline{P}$, $\overline{AB}$.

Since the displacement from B to A is the opposite of a displacement from A to B, we write

$$\overline{BA} = -\overline{AB}$$

and take vectors in opposite senses to have opposite signs. Since two successive displacements of a point from A to B and from B to C produce the same result as a single displacement from A to C, we say that the vector AC is equal to the sum of the vectors AB, BC and write



$$\overline{AC} = \overline{AB} + \overline{BC} \quad \text{......(1),}$$

and further, if $A, B, C \ldots K, L$ are any set of points

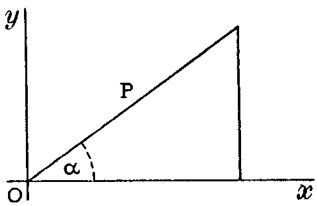
$$\overline{AL} = \overline{AB} + \overline{BC} + \ldots + \overline{KL} \quad \text{...............(2).}$$

Vectors in general are not localized; thus we may have a displacement of an assigned length in an assigned direction and sense but its locality not specified. In such a case all equal and parallel lines in the same sense will represent the same vector. On the other hand, vectors may be localized, either at a point, e.g. the *velocity* of a particle; or in a line, as for example a *force* whose line of action (but not point of application) is specified.

2·2. Composition of Vectors. A single vector which is equivalent to two or more vectors is called their **resultant**, and they are called the **components** of the resultant. Vectors are compounded by geometrical addition as indicated in formulae (1) and (2) of the last Article.

A vector can be resolved into two components in assigned directions in the same plane; for if AC be the vector, and through A, C two lines are drawn in the assigned directions meetings in B, then AB, BC are the components required.

When a vector is resolved into two components in directions at right angles to one another, each component is called the **resolved part of the vector** in the direction specified. Thus if a vector $\mathbf{P}$ makes an angle α with a given direction Ox, the resolved parts of $\mathbf{P}$ in the direction Ox and in the perpendicular direction Oy are



$$\mathbf{P}\cos\alpha \text{ and } \mathbf{P}\sin\alpha.$$

2·3. Since the algebraical sum of the orthogonal projections on any straight line of the sides of a closed polygon is zero, it follows that the orthogonal projection of the resultant of a number of vectors is equal to the algebraical sum of the projections of the component vectors.

2·4. Analytical Method. To compound n vectors $\mathbf{P}_1, \mathbf{P}_2 \ldots \mathbf{P}_n$. Let the vectors make angles $\alpha_1, \alpha_2 \ldots \alpha_n$ with an axis Ox. Each vector may be resolved into two components, one in the direction Ox and the other in the perpendicular direction Oy. The components in direction Ox are equivalent to a single vector

$$\mathbf{X} = \mathbf{P}_1\cos\alpha_1 + \mathbf{P}_2\cos\alpha_2 + \ldots + \mathbf{P}_n\cos\alpha_n = \Sigma\,(\mathbf{P}\cos\alpha),$$

and the components in direction Oy are equivalent to a single vector

$$\mathbf{Y} = \mathbf{P}_1 \sin \alpha_1 + \mathbf{P}_2 \sin \alpha_2 + \ldots + \mathbf{P}_n \sin \alpha_n = \Sigma (\mathbf{P} \sin \alpha).$$

The two vectors $\mathbf{X}$, $\mathbf{Y}$ can now be compounded into a single vector $\mathbf{R}$ making an angle θ with Ox, such that

$$\mathbf{R} \cos \theta = \mathbf{X} \text{ and } \mathbf{R} \sin \theta = \mathbf{Y},$$

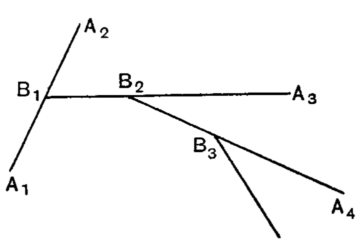
and therefore $R^2 = X^2 + Y^2$ and $\tan \theta = Y/X$ ....(1).

2·5. Vectors may be multiplied and divided by scalar numbers. Thus if we take n equal vectors $\overline{AB}$ and compound them together we get a vector $\overline{AC}$, such that $\overline{AC} = n\overline{AB}$; and, conversely, $\overline{AB} = \frac{1}{n}\overline{AC}$.

Note that relations of the form $\overline{AC} = n\overline{AB}$, or $p\overline{AB} + q\overline{AC} = 0$ imply that the points A, B, C are in the same straight line.

2·6. Centroids or Mean Centres. If $m_1, m_2, m_3 \ldots m_n$ be a set of scalar magnitudes associated with a set of points $A_1, A_2, A_3 \ldots A_n$, the centroid or mean centre of the points for the given magnitudes is the point obtained by the following process: Divide the line A_1A_2 at B_1 so that $m_1 A_1B_1 = m_2 B_1A_2$; divide B_1A_3 at B_2 so that $(m_1 + m_2) B_1B_2 = m_3 B_2A_3$; divide B_2A_4 at B_3 so that $(m_1 + m_2 + m_3) B_2B_3 = m_4 B_3A_4$. Proceed in this way until all the points have been connected then the last point of division B_{n-1}, usually denoted by the letter G, is called the *centroid* or *mean centre*.

2·61. In order to shew that this process leads in general to a unique point, i.e. that the point determined by the process is independent of the order in which the points $A_1, A_2 \ldots A_n$ are joined, we shall first prove that



$$m_1\overline{A_1G} + m_2\overline{A_2G} + \ldots + m_n\overline{A_nG} = 0 \ \ldots(1).$$

Assume that this formula is true for the first r points, i.e. that

$$m_1\overline{A_1B_{r-1}} + m_2\overline{A_2B_{r-1}} + \ldots + m_r\overline{A_rB_{r-1}} = 0.$$

Now the next step in the process is to divide $B_{r-1}A_{r+1}$ at B_r so that

$$(m_1+m_2+\ldots+m_r)\,\overline{B_{r-1}B_r}=m_{r+1}\overline{B_rA_{r+1}},$$

therefore by adding the last two lines

$$m_1\overline{A_1B_r}+m_2\overline{A_2B_r}+\ldots+m_r\overline{A_rB_r}+m_{r+1}\overline{A_{r+1}B_r}=0.$$

It follows that if the formula (1) is true for r points it is also true for $r+1$; but it is true for two points, since, by hypothesis, $m_1\overline{A_1B_1}+m_2\overline{A_2B_1}=0$. Therefore the formula (1) is true for any number of points.

Now if by taking the points in a different order we arrive at a centroid G' we can shew similarly that

$$m_1\overline{A_1G'}+m_2\overline{A_2G'}+\ldots+m_n\overline{A_nG'}=0\;\ldots\ldots\ldots(2);$$

and by subtracting (1) from (2) we get

$$(m_1+m_2+\ldots+m_n)\,\overline{GG'}=0.$$

Hence G' must coincide with G unless $m_1+m_2+\ldots+m_n=0$. In the latter case there is no centroid at a finite distance, because the last step in the process of finding the centroid consists in dividing a line in the ratio $m_1+m_2+\ldots+m_{n-1}:m_n$, i.e. in the ratio $1:-1$.

2·7. Centroid Method of Compounding Vectors. To shew, with the notation of the last Article, that, if O be any other point, the resultant of n vectors $m_1\overline{OA_1}$, $m_2\overline{OA_2}\ldots m_n\overline{OA_n}$ is $(m_1+m_2+\ldots+m_n)\,\overline{OG}$, where G is the centroid of the points $A_1, A_2\ldots A_n$ for the magnitudes $m_1, m_2\ldots m_n$.

This follows at once by substituting

$$\overline{OA_1}=\overline{OG}+\overline{GA_1},\;\overline{OA_2}=\overline{OG}+\overline{GA_2},\text{ etc.,}$$

so that

$$m_1\overline{OA_1}+m_2\overline{OA_2}+\ldots+m_n\overline{OA_n}$$
$$=(m_1+m_2+\ldots+m_n)\,\overline{OG}+(m_1\overline{GA_1}+m_2\overline{GA_2}+\ldots+m_n\overline{GA_n});$$

and by **2·61** (1) the sum of the terms in the last bracket is zero, therefore

$$m_1\overline{OA_1}+m_2\overline{OA_2}+\ldots+m_n\overline{OA_n}=(m_1+m_2+\ldots+m_n)\,\overline{OG}.$$

2·8. When reference is made to the centroid of a set of points without mention of any associated magnitudes it is understood that the magnitudes are equal; thus the centroid of a triangle ABC is a point G such that

$$\overline{AG} + \overline{BG} + \overline{CG} = 0.$$

2·9. It may be noticed that if **P**, **Q**, **R** are vectors in the lines OA, OB, OC then the resultant vector is

$$\left(\frac{P}{OA} + \frac{Q}{OB} + \frac{R}{OC}\right)\overline{OG},$$

where G is the centroid of the points A, B, C for the magnitude P/OA, Q/OB, R/OC; for a vector **P** is the same as $\dfrac{P}{OA}\overline{OA}$.

EXAMPLES

1. Prove that, if G is the middle point of AB and G' is the middle point of $A'B'$, then $\overline{AA'} + \overline{BB'} = 2\overline{GG'}$.

2. Prove that, if G is the centroid of n points $A_1, A_2 \ldots A_n$, and G' is the centroid of n points $B_1, B_2 \ldots B_n$, then

$$\overline{A_1B_1} + \overline{A_2B_2} + \ldots + \overline{A_nB_n} = n\overline{GG'}.$$

3. Prove that, if H is the orthocentre and O is the circumcentre of a triangle ABC, then

$$\overline{AH}\tan A + \overline{BH}\tan B + \overline{CH}\tan C = 0,$$

and

$$\overline{AO}\sin 2A + \overline{BO}\sin 2B + \overline{CO}\sin 2C = 0.$$

4. Shew that, if $m\overline{OA} + n\overline{OB} + p\overline{OC} = 0$ and $m + n + p = 0$, then the points A, B, C are collinear.