ELEMENTS

OF

ALGEBRA.

PART I.

Containing the Analysis *of* Determinate Quantities.

SECTION I.

Of the different Methods *of calculating* Simple Quantities.

CHAP. I.

Of Mathematics *in general.*

ARTICLE I.

WHATEVER is capable of increase or diminution, is called *magnitude*, or *quantity*.

A sum of money therefore is a quantity, since we may increase it or diminish it. It is the same with a weight, and other things of this nature.

2. From this definition, it is evident, that the different kinds of magnitude must be so various, as to render it difficult to enumerate them: and this is the origin of the different branches of the Mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity;* or, the science which investigates the means of measuring quantity.

3. Now, we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money,

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as a louis, a crown, a ducat, or some other coin, and shew how many of these pieces are contained in the given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight; for example, a pound, an ounce, &c. and then shew how many times one of these weights is contained in that which we are endeavouring to ascertain. If we wished to measure any length or extension, we should make use of some known length, such as a foot.

4. So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit;* then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.

5. From this it appears, that all magnitudes may be expressed by numbers; and that the foundation of all the Mathematical Sciences must be laid in a complete treatise on the science of Numbers, and in an accurate examination of the different possible methods of calculation.

This fundamental part of mathematics is called Analysis, or Algebra *.

6. In Algebra then we consider only numbers, which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of the mathematics.

7. Arithmetic treats of numbers in particular, and is the *science of numbers properly so called;* but this science extends only to certain methods of calculation which occur in common practice: Algebra, on the contrary, comprehends in general all the cases that can exist in the doctrine and calculation of numbers.

* Several mathematical writers make a distinction between *Analysis* and *Algebra*. By the term *Analysis*, they understand the method of determining those general rules, which assist the understanding in all mathematical investigations; and by *Algebra*, the instrument which this method employs for accomplishing that end. This is the definition given by M. Bezout in the preface to his Algebra. F. T.

CHAP. II.

Explanation of the Signs + Plus *and* − Minus.

8. When we have to add one given number to another, this is indicated by the sign +, which is placed before the second number, and is read *plus*. Thus 5 + 3 signifies that we must add 3 to the number 5, in which case, every one knows that the result is 8; in the same manner 12 + 7 make 19; 25 + 16 make 41; the sum of 25 + 41 is 66, &c.

9. We also make use of the same sign + *plus*, to connect several numbers together; for example, 7 + 5 + 9 signifies that to the number 7 we must add 5, and also 9, which make 21. The reader will therefore understand what is meant by

$$8 + 5 + 13 + 11 + 1 + 3 + 10,$$

viz. the sum of all these numbers, which is 51.

10. All this is evident; and we have only to mention, that in Algebra, in order to generalise numbers, we represent them by letters, as a, b, c, d, &c. Thus, the expression $a + b$, signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great, or very small. In the same manner, $f + m + b + x$, signifies the sum of the numbers represented by these four letters.

If we know therefore the numbers that are represented by letters, we shall at all times be able to find, by arithmetic, the sum or value of such expressions.

11. When it is required, on the contrary, to subtract one given number from another, this operation is denoted by the sign −, which signifies *minus*, and is placed before the number to be subtracted: thus, 8 − 5 signifies that the number 5 is to be taken from the number 8; which being done, there remain 3. In like manner 12 − 7 is the same as 5; and 20 − 14 is the same as 6, &c.

12. Sometimes also we may have several numbers to subtract from a single one; as, for instance, 50 − 1 − 3 − 5 − 7 − 9. This signifies, first, take 1 from 50, and there remain 49; take 3 from that remainder, and there will remain 46; take away 5, and 41 remain; take away 7, and 34 remain; lastly, from that take 9, and there remain 25: this last remainder is the value of the expression. But as the numbers 1, 3, 5, 7, 9, are all to be subtracted, it is the

same thing if we subtract their sum, which is 25, at once from 50, and the remainder will be 25 as before.

13. It is also easy to determine the value of similar expressions, in which both the signs + *plus* and — *minus* are found. For example;

$$12 - 3 - 5 + 2 - 1 \text{ is the same as } 5.$$

We have only to collect separately the sum of the numbers that have the sign + before them, and subtract from it the sum of those that have the sign —. Thus, the sum of 12 and 2 is 14; and that of 3, 5, and 1, is 9; hence 9 being taken from 14, there remain 5.

14. It will be perceived, from these examples, that the order in which we write the numbers is perfectly indifferent and arbitrary, provided the proper sign of each be preserved. We might with equal propriety have arranged the expression in the preceding article thus; $12 + 2 - 5 - 3 - 1$, or $2 - 1 - 3 - 5 + 12$, or $2 + 12 - 3 - 1 - 5$, or in still different orders; where it must be observed, that in the arrangement first proposed, the sign + is supposed to be placed before the number 12.

15. It will not be attended with any more difficulty if, in order to generalise these operations, we make use of letters instead of real numbers. It is evident, for example, that

$$a - b - c + d - e,$$

signifies that we have numbers expressed by a and d, and that from these numbers, or from their sum, we must subtract the numbers expressed by the letters b, c, e, which have before them the sign —.

16. Hence it is absolutely necessary to consider what sign is prefixed to each number: for in Algebra, simple quantities are numbers considered with regard to the signs which precede, or affect them. Farther, we call those *positive quantities*, before which the sign + is found; and those are called *negative quantities*, which are affected by the sign —.

17. The manner in which we generally calculate a person's property, is an apt illustration of what has just been said. For we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign —. Thus, when it is said of any one that he has 100 crowns, but owes 50, this means that his real possession amounts to $100 - 50$; or, which is the same thing, $+ 100 - 50$, that is to say, 50.

18. Since negative numbers may be considered as debts, because positive numbers represent real possessions, we

may say that negative numbers are less than nothing. Thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before.

19. In the same manner, therefore, as positive numbers are incontestably greater than nothing, negative numbers are less than nothing. Now, we obtain positive numbers by adding 1 to 0, that is to say, 1 to nothing; and by continuing always to increase thus from unity. This is the origin of the series of numbers called *natural numbers;* the following being the leading terms of this series:

$0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10,$
and so on to infinity.

But if, instead of continuing this series by successive additions, we continued it in the opposite direction, by perpetually subtracting unity, we should have the following series of negative numbers:

$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10,$
and so on to infinity.

20. All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers*, which consequently are either greater or less than nothing. We call them *integers*, to distinguish them from fractions, and from several other kinds of numbers, of which we shall hereafter speak. For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be, between 49 and 50, an infinity of intermediate numbers, all greater than 49, and yet all less than 50. We need only imagine two lines, one 50 feet, the other 49 feet long, and it is evident that an infinite number of lines may be drawn, all longer than 49 feet, and yet shorter than 50.

21. It is of the utmost importance through the whole of Algebra, that a precise idea should be formed of those negative quantities, about which we have been speaking. I shall, however, content myself with remarking here, that all such expressions as

$$+1-1,\ +2-2,\ +3-3,\ +4-4, \text{ \&c.}$$

are equal to 0, or nothing. And that

$$+2-5 \text{ is equal to } -3:$$

for if a person has 2 crowns, and owes 5, he has not only nothing, but still owes 3 crowns. In the same manner,

$7-12$ is equal to -5, and $25-40$ is equal to -15.

22. The same observations hold true, when, to make the expression more general, letters are used instead of numbers;

thus 0, or nothing, will always be the value of $+a-a$; but if we wish to know the value of $+a-b$, two cases are to be considered.

The first is when a is greater than b; b must then be subtracted from a, and the remainder (before which is placed, or understood to be placed, the sign $+$) shews the value sought.

The second case is that in which a is less than b: here a is to be subtracted from b, and the remainder being made negative, by placing before it the sign $-$, will be the value sought.

CHAP. III.

Of the Multiplication *of* Simple Quantities.

23. When there are two or more equal numbers to be added together, the expression of their sum may be abridged: for example,

$a + a$ is the same with $2 \times a$,
$a + a + a$ - - - - - - $3 \times a$,
$a + a + a + a$ - - - $4 \times a$, and so on; where $\times$ is the sign of multiplication. In this manner we may form an idea of multiplication; and it is to be observed that,

$2 \times a$ signifies 2 times, or twice a
$3 \times a$ - - - - 3 times, or thrice a
$4 \times a$ - - - - 4 times a, &c.

24. If therefore a number expressed by a letter is to be multiplied by any other number, we simply put that number before the letter, thus;

a multiplied by 20 is expressed by $20a$, and
b multiplied by 30 is expressed by $30b$, &c.

It is evident, also, that c taken once, or $1c$, is the same as c.

25. Farther, it is extremely easy to multiply such products again by other numbers; for example:

2 times, or twice $3a$, makes $6a$
3 times, or thrice $4b$, makes $12b$
5 times $7x$ makes $35x$.

and these products may be still multiplied by other numbers at pleasure.

26. When the number by which we are to multiply is also represented by a letter, we place it immediately before the other letter; thus, in multiplying b by a, the product is

written ab; and pq will be the product of the multiplication of the number q by p. Also, if we multiply this pq again by a, we shall obtain apq.

27. It may be farther remarked here, that the order in which the letters are joined together is indifferent; thus ab is the same thing as ba; for b multiplied by a is the same as a multiplied by b. To understand this, we have only to substitute, for a and b, known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3.

28. It will not be difficult to perceive, that when we substitute numbers for letters joined together, in the manner we have described, they cannot be written in the same way by putting them one after the other. For if we were to write 34 for 3 times 4, we should have 34 and not 12. When therefore it is required to multiply common numbers, we must separate them by the sign $\times$, or by a point: thus, 3×4, or 3.4, signifies 3 times 4; that is, 12. So, 1×2 is equal to 2; and $1 \times 2 \times 3$ makes 6. In like manner, $1 \times 2 \times 3 \times 4 \times 56$ makes 1344; and $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$ is equal to 3628800, &c.

29. In the same manner we may discover the value of an expression of this form, $5.7.8.abcd$. It shews that 5 must be multiplied by 7, and that this product is to be again multiplied by 8; that we are then to multiply this product of the three numbers by a, next by b, then by c, and lastly by d. It may be observed, also, that instead of 5.7.8, we may write its value, 280; for we obtain this number when we multiply the product of 5 by 7, or 35, by 8.

30. The results which arise from the multiplication of two or more numbers are called *products;* and the numbers, or individual letters, are called *factors.*

31. Hitherto we have considered only positive numbers, and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+a$ by $+b$ must necessarily give $+ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down

as a rule, that + by + makes + or *plus;* and that, on the contrary, + by −, or − by +, gives −, or *minus*.

33. It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by − produces +, that is, the same as + multiplied by + *.

* A further illustration of this rule is generally given by algebraists as follows:

First, we know that $+a$ multiplied by $+b$ gives the product $+ab$; and if $+a$ be multiplied by a quantity less than b, as $b-c$, the product must necessarily be less than ab; in short, from ab we must subtract the product of a, multiplied by c; hence $a \times (b-c)$ must be expressed by $ab-ac$; therefore it follows that $a \times -c$ gives the product $-ac$.

If now we consider the product arising from the multiplication of the two quantities $(a-b)$, and $(c-d)$, we know that it is less than that of $(a-b) \times c$, or of $ac - bc$; in short, from this product we must subtract that of $(a-b) \times d$; but the product $(a-b) \times (c-d)$ becomes $ac - bc - ad$, together with the product of $-b \times -d$ annexed; and the question is only what sign we must employ for this purpose, whether + or −. Now we have seen that from the product $ac - bc$ we must subtract the product of $(a-b) \times d$, that is, we must subtract a quantity less than ad; we have therefore subtracted already too much by the quantity bd; this product must therefore be added; that is, it must have the sign + prefixed; hence we see that $-b \times -d$ gives $+bd$ for a product; or − *minus* multiplied by − *minus* gives + *plus*. See Art. 273, 274.

Multiplication has been erroneously called a compendious method of performing addition: whereas it is the taking, or repeating of one given number as many times, as the number by which it is to be multiplied, contains units. Thus, 9×3 means that 9 is to be taken 3 times, or that the measure of multiplication is 3; again, $9 \times \frac{1}{2}$ means that 9 is to be taken half a time, or that the measure of multiplication is $\frac{1}{2}$. In multiplication there are two factors, which are sometimes called the multiplicand and the multiplier. These, it is evident, may reciprocally change places, and the product will be still the same: for $9 \times 3 = 3 \times 9$, and $9 \times \frac{1}{2} = \frac{1}{2} \times 9$. Hence it appears, that numbers may be diminished by multiplication, as well as increased, in any given ratio, which is wholly inconsistent with

34. The rules which we have explained are expressed more briefly as follows:

Like signs, multiplied together, give $+$; unlike or contrary signs give $-$. Thus, when it is required to multiply the following numbers; $+a$, $-b$, $-c$, $+d$; we have first $+a$ multiplied by $-b$, which makes $-ab$; this by $-c$, gives $+abc$; and this by $+d$, gives $+abcd$.

35. The difficulties with respect to the signs being removed, we have only to shew how to multiply numbers that are themselves products. If we were, for instance, to multiply the number ab by the number cd, the product would be $abcd$, and it is obtained by multiplying first ab by c, and then the result of that multiplication by d. Or, if we had to multiply 36 by 12; since 12 is equal to 3 times 4, we

the nature of Addition; for $9 \times \frac{1}{2} = 4\frac{1}{2}$, $9 \times \frac{1}{9} = 1$, $9 \times \frac{1}{900} = \frac{1}{100}$, &c. The same will be found true with respect to algebraic quantities; $a \times b = ab$, $-9 \times 3 = -27$, that is, 9 negative integers multiplied by 3, or taken 3 times, are equal to -27, because the measure of multiplication is 3. In the same manner, by inverting the factors, 3 positive integers multiplied by -9, or taken 9 times negatively, must give the same result. Therefore a positive quantity taken negatively, or a negative quantity taken positively, gives a negative product.

From these considerations, we may illustrate the present subject in a different way, and shew, that the product of two negative quantities must be positive. First, algebraic quantities may be considered as a series of numbers increasing in any ratio, on each side of nothing, to infinity. Let us assume a small part only of such a series for the present purpose, in which the ratio is unity, and let us multiply every term of it by -2.

5,	4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,	−5,
−2,	−2,	−2,	−2,	−2,	−2,	−2,	−2,	−2,	−2,	−2,
−10,	−8,	−6,	−4,	−2,	0,	+2,	+4,	+6,	+8,	+10.

Here, of course, we find the series inverted, and the ratio doubled. Farther, in order to illustrate the subject, we may consider the ratio of a series of fractions between 1 and 0, as indefinitely small, till the last term being multiplied by -2, the product would be equal to 0. If, after this, the multiplier having passed the middle term, 0, be multiplied into any negative term, however small, between 0 and -1, on the other side of the series, the product, it is evident, must be positive, otherwise the series could not go on. Hence it appears, that the taking of a negative quantity negatively destroys the very property of negation, and is the conversion of negative into positive numbers. So that if $+ \times - = -$, it necessarily follows that $- \times -$ must give a contrary product, that is, $+$. See Art. 176, 177.

should only multiply 36 first by 3, and then the product 108 by 4, in order to have the whole product of the multiplication of 12 by 36, which is consequently 432.

36. But if we wished to multiply $5ab$ by $3cd$, we might write $3cd \times 5ab$. However, as in the present instance the order of the numbers to be multiplied is indifferent, it will be better, as is also the custom, to place the common numbers before the letters, and to express the product thus: $5 \times 3abcd$, or $15abcd$; since 5 times 3 is 15.

So if we had to multiply $12pqr$ by $7xy$, we should obtain $12 \times 7pqrxy$, or $84pqrxy$.

CHAP. IV.

Of the Nature of whole Numbers, *or* Integers, *with respect to their* Factors.

37. We have observed that a product is generated by the multiplication of two or more numbers together, and that these numbers are called *factors*. Thus, the numbers a, b, c, d, are the factors of the product $abcd$.

38. If, therefore, we consider all whole numbers as products of two or more numbers multiplied together, we shall soon find that some of them cannot result from such a multiplication, and consequently have not any factors; while others may be the products of two or more numbers multiplied together, and may consequently have two or more factors. Thus 4 is produced by 2×2; 6 by 2×3; 8 by $2 \times 2 \times 2$; 27 by $3 \times 3 \times 3$; and 10 by 2×5, &c.

39. But on the other hand, the numbers 2, 3, 5, 7, 11, 13, 17, &c. cannot be represented in the same manner by factors, unless for that purpose we make use of unity, and represent 2, for instance, by 1×2. But the numbers which are multiplied by 1 remaining the same, it is not proper to reckon unity as a factor.

All numbers, therefore, such as 2, 3, 5, 7, 11, 13, 17, &c. which cannot be represented by factors, are called *simple*, or *prime numbers*; whereas others, as 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, &c. which may be represented by factors, are called *composite numbers*.

40. *Simple* or *prime numbers* deserve therefore particular attention, since they do not result from the mul-