MATHEMATICAL AND PHYSICAL PAPERS.

NOTE ON CERTAIN FORMULÆ IN THE CALCULUS OF OPERATIONS. (In a letter to Prof. TAIT.)

[From the *Proceedings of the Royal Society of Edinburgh*, XI, pp. 101–2.]

"*January* 14*th*, 1876.

"FORMULÆ like those you sent me* are readily *suggested* by supposing the function operated on to be of the form $\Sigma A x^\alpha$, or say, for shortness, x^α, with the understanding that no transformations are to be made which are not equally valid for $\Sigma A x^\alpha$.

Thus

$$\left(\frac{d}{dx}\,x\,\frac{d}{dx}\right)^n x^\alpha = \alpha^2(\alpha-1)^2 \ldots (\alpha-n+1)^2 x^{\alpha-n}$$

$$= \alpha(\alpha-1)\ldots(\alpha-n+1)\left(\frac{d}{dx}\right)^n x^\alpha$$

$$= \left(\frac{d}{dx}\right)^n x^n \left(\frac{d}{dx}\right)^n x^\alpha\,;$$

and

$$\left(x\,\frac{d}{dx}\,x\right)^n x^\alpha = (\alpha+1)(\alpha+2)\ldots(\alpha+n)\,x^{\alpha+n}$$

$$= (\alpha+n)(\alpha+n-1)\ldots(\alpha+1)\,x^{\alpha+n}$$

$$= x^n\left(\frac{d}{dx}\right)^n x^n x^\alpha\,.$$

The direct transformation may readily be effected by noticing, in the first instance, that any two operations of the form

$$x^{-m+1}\frac{d}{dx}x^m$$

* See *ante* [*Proc. R. S. Edin.*], p. 95.

are convertible. We find, in fact,

$$x^{-m+1}\frac{d}{dx}x^m\,.\,x^{-n+1}\frac{d}{dx}x^n = x^2\left(\frac{d}{dx}\right)^2 + (m+n+1)\,x\frac{d}{dx} + mn,$$

into which m and n enter symmetrically.

Replacing the operations in the left hand member of the first formula by convertible operations, which will be separated by points, we find

$$\frac{d}{dx}x\frac{d}{dx} = x^{-1}\times x\frac{d}{dx}\,.\,x\frac{d}{dx},$$

$$\frac{d}{dx}x\frac{d}{dx}x^{-1} = x^{-2}\times x^2\frac{d}{dx}x^{-1}\,.\,x^2\frac{d}{dx}x^{-1},$$

and so on. Hence

$$\left(\frac{d}{dx}x\frac{d}{dx}\right)^n = x^{-n}\left(x^n\frac{d}{dx}x^{-n+1}\right)^2\left(x^{n-1}\frac{d}{dx}x^{-n+2}\right)^2\ldots\left(x\frac{d}{dx}\right)^2,$$

$$= x^{-n}\left\{x^n\frac{d}{dx}x^{-n+1}\,.\,x^{n-1}\frac{d}{dx}x^{-n+2}\ldots x\frac{d}{dx}\right\}^2,$$

$$= x^{-n}\left\{x^n\left(\frac{d}{dx}\right)^n\right\}^2 = \left(\frac{d}{dx}\right)^n x^n\left(\frac{d}{dx}\right)^n.$$

Again,

$$x\frac{d}{dx}x = x\times\frac{d}{dx}x,$$

$$x\frac{d}{dx}x^2 = x^2\times x^{-1}\frac{d}{dx}x^2,$$

and so on. Hence

$$\left(x\frac{d}{dx}x\right)^n = x^n\times x^{-n+1}\frac{d}{dx}x^n\,.\,x^{-n+2}\frac{d}{dx}x^{n-1}\ldots\frac{d}{dx}x,$$

$$= x^n\times\frac{d}{dx}x\,.\,x^{-1}\frac{d}{dx}x^2\ldots x^{-n+1}\frac{d}{dx}x^n,$$

$$= x^n\left(\frac{d}{dx}\right)^n x^n.\text{''}$$

AN EXPERIMENT ON ELECTRO-MAGNETIC ROTATION.

By W. SPOTTISWOODE, TREAS.R.S. [Extract.]

[From the *Proceedings of the Royal Society*, XXIV, *Feb.* 24, 1876, pp. 403—407.]

[THE experiment is on the rotation and spirality assumed in a magnetic field by the luminous electric discharge through a gas.]

The following explanation of the phenomenon is due to Prof. Stokes, from whose correspondence it is substantially taken. The mathematical solution, although only roughly approximate, is perhaps still quite sufficient to give the general character of the experimental results *.

The magnetic field will be supposed uniform, and the lines of force parallel straight lines from pole to pole. The path of the current when undisturbed is also a straight line from pole to pole. In such a condition of things, everything being symmetrical, no rotation would take place. But if through any local circumstance, as in the experiment in air, or through heating of the chamber as in the exhausted tube, or otherwise, the path of the current be distorted and displaced, then each element will be subject to the action of two forces. To estimate these, let ds be an element of the path, with rectangular components dx, dy, dz, C the strength of the current, and R the magnetic force with components X, Y, Z, which in the first instance will be treated generally. Then one force will be that tending to impel the current in the direction of the axes respectively, and may be expressed by

$$C(Ydz - Zdy)/ds, \quad C(Zdx - Xdz)/ds, \quad C(Xdy - Ydx)/ds.$$

Besides this, there will be the tendency of the current to follow

[* The electric discharge is considered as represented by a current in a flexible inextensible conducting thread; see p. 243 *supra*. On the catenary, cf. Larmor, *Proc. Lond. Math. Soc.* 1884, p. 170; and on the physical problem, cf. J. J. Thomson, *Conduction of Electricity through Gases*, Ch. IV.]

the shortest path so as to diminish the resistance. Representing this as a tension τ, the components at one end of ds will be

$$-\tau dx/ds, \quad -\tau dy/ds, \quad -\tau dz/ds,$$

and those at the other

$$(\tau dx/ds) + d(\tau dx/ds), \ldots,$$

the algebraical sums of which are

$$d(\tau dx/ds), \quad d(\tau dy/ds), \quad d(\tau dz/ds),$$

and the equations of equilibrium then become

$$C(Ydz - Zdy) + d(\tau dx/ds) = 0 \quad\ldots\ldots\ldots\ldots(1),$$

$$C(Zdx - Xdz) + d(\tau dy/ds) = 0 \quad\ldots\ldots\ldots\ldots(2),$$

$$C(Xdy - Ydx) + d(\tau dz/ds) = 0 \quad\ldots\ldots\ldots\ldots(3);$$

taking s as the independent variable and multiplying by dx/ds dy/ds, dz/ds respectively, and adding, we obtain $d\tau = 0$, or $\tau =$ constant. Again, multiplying by X, Y, Z and adding we obtain

$$Xd^2x/ds^2 + Yd^2y/ds^2 + Zd^2z/ds^2 = 0 \quad\ldots\ldots\ldots\ldots(4),$$

which expresses that the absolute normal (or normal in the osculating plane) is perpendicular to the resultant magnetic force.

In the case of a uniform tint, X, Y, Z will be constant. Integrating (4) and putting i for the angle between the tangent and the lines of magnetic force, we find

$$Xdx + Ydy + Zdz = Rds\cos i,$$

so that the tangent line is inclined at a constant angle to the line joining the poles.

Again, the following combinations,

$$(2)\,dz - (3)\,dy = 0, \quad (3)\,dx - (1)\,dz = 0, \quad (1)\,dy - (2)\,dx = 0$$

give

$$Cdx(Xdx + \ldots) - CXds^2 + \tau\left(\frac{dz}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2z}{ds^2}\right)ds^2 = 0, \text{ \&c.},$$

or

$$C(R\cos i\,dx - Xds) + \tau\left(\frac{dz}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2z}{ds^2}\right)ds = 0, \text{ \&c.}$$

Transposing, squaring, and adding, and putting ρ for the radius of curvature, we obtain

$$C^2R^2\sin^2 i = \tau^2/\rho^2, \quad \text{or} \quad \rho = \tau/CR\sin i,$$

which is constant. The curve is therefore a helix. Also the radius of curvature of the projection of the curve on a plane perpendicular to the axis will be $\rho \sin^2 i$, viz. $= \tau \sin i/CR$.

"The value of τ depends doubtless on the nature and pressure of the gas, and perhaps also on the current; but it must be the same for equal values of C of opposite signs. Hence the handedness of the helix will be reversed by reversing either the current or the magnetic polarity. If the left-hand magnetic pole be north (*i.e.* austral, or north-pointing), and the left-hand terminal positive, the helix will be right-handed."

The general nature of the phenomenon may therefore now be described as follows:—"First, we have the bright spark of no sensible duration which strikes nearly in a straight line between the terminals. This opens a path for a continuous discharge, which being nearly in a condition of equilibrium, though an unstable one, remains a short time without much change of place. Then it moves rapidly to its position of equilibrium, the surface which is its locus forming the sheet. Then it remains in its position of equilibrium during the greater part of the discharge, approaching the axis again as the discharge falls, so that its equilibrium position is not so far from the axis. Thus we see two bright curves corresponding to the two positions of approximate rest united by a less bright sheet, the first curve being nearly a straight line, and the second nearly a helix traced on a cylinder of which the former line is a generating line.

"It was noticed that the sheet projected a little beyond the helix. This may be explained by considering that at first the discharge is more powerful than can be maintained, so that the curve reaches a little beyond the distance that can be maintained."

The appearance of the discharge when viewed in a revolving mirror (except the projection beyond the sheet, the illumination of which was too feeble to be observed) confirmed the above remarks.

ON THE FOCI OF LINES SEEN THROUGH A CRYSTALLINE PLATE.

[From the *Proceedings of the Royal Society*, XXVI, pp. 386—401, *June* 21, 1877.]

AT the Soirée of the Royal Society on the 25th of April Mr Sorby showed me the method he had recently devised for discriminating between minerals by focusing a microscope over a delicate image of cross lines, which image was viewed, first directly, and then through a crystalline plate, having previously been adjusted to be at the distance of the lower surface of the plate. With glass and singly refracting substances the alteration of the focus produced by the interposition of the plate affords a measure of its refractive index. But with a plate cut from a doubly refracting crystal, not only is there more than one focal distance, but for one at least of the pencils there is (except in special cases) no true focus, but the foci of the two systems of cross lines are found at two different depths, or else there is no sharply defined image at all, according to the orientation of the lines relatively to lines fixed in the crystalline plate. Moreover the result obtained on applying the formula which, for a singly refracting plate, gives the refractive index from the measured displacement of the focus is often widely different from what is known to be the refractive index of the crystal, for the pencil under examination, in a direction perpendicular to the plate.

The phenomena will be described in detail by Mr Sorby in his own paper. My object is to show how they flow from the known laws of double refraction, as consequences of which they will necessarily come under review*.

[* It seemed pretty certain that some of the phenomena must have been noticed before, though I am not aware that they have been described, or their theory worked out in any detail. I find that Prof. Clifton has been in the habit of using an instrument somewhat similar to Mr Sorby's, which was procured several years ago for the Museum of the University of Oxford, and that he was familiar with

The simplest case is that of a uniaxal crystal, such as Iceland spar, cut perpendicular to its axis. As regards the ordinary ray, a plate cut from a uniaxal crystal, in whatever direction, behaves, of course, like a plate of glass, so far as focusing is concerned, and the index obtained is the true ordinary index. To find what takes place as regards the extraordinary ray, we must have recourse to Huyghens's construction.

Let O be any point in the further surface of the crystalline plate, OA perpendicular to the surface the direction of the axis, OP the direction of any extraordinary ray. Let the plane of the

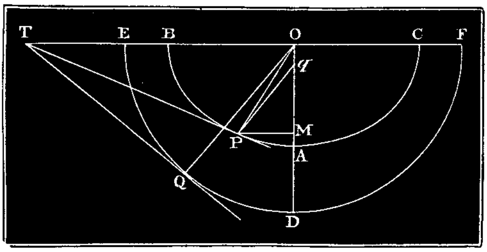

Fig. 1.

paper be the plane of incidence, AOP; take OA to represent the velocity of propagation (a) within the crystal in the direction of the axis, and OD in OA produced to represent the velocity of propagation (unity) in air. With O as centre, construct the half-spheroid, BAC, which is the extraordinary sheet of the wave-surface, and the hemisphere EDF representing the wave into which a disturbance emanating from O would have spread in air

such things as the low apparent index of calcite for the extraordinary pencil nearly in the direction of the axis, and the astigmatism in general of a pencil refracted across a plate otherwise than by ordinary refraction; and, further, that he utilized these phenomena for the instruction of students as to the general form of the wave-surface. No one, however, so far as I know, before Mr Sorby, had applied the phenomena to the practical discrimination of minerals, or had worked them out quantitatively and in detail; and it is my desire to complete the subject, by supplying the mathematical theory, that must be my excuse for offering to the Society an investigation which in itself consists merely in easy deductions from well-known principles.

It is perhaps hardly necessary to refer to a paper by Dr Quincke in *Poggendorff's Annalen* (1862, Vol. xxvii, p. 563), containing some elaborate observations on the focal lines formed within a doubly refracting plate, as his experiments were made in a very different manner from those of Mr Sorby, and with a totally different object in view. October, 1877.]

in a unit of time, and let OB or OC be denoted by c. Let OP cut the half-spheroid in P. At P draw a tangent plane to the spheroid, the trace of which on the surface of the crystal is projected in T; and through the trace T draw a tangent plane to the hemisphere, touching it in Q, and join OQ. Then if an extraordinary ray travel within the crystal in the direction OP, the refracted ray to which it will give rise will travel in a direction parallel to OQ. Hence if we now take OP to denote the whole path of the ray within the plate, and draw Pq parallel to QO, cutting OA in q, the ray OP, after refraction at P, will proceed as if it came from q. Hence the limiting position of q, as P moves up to A, will be the geometrical focus, after refraction, of a small pencil proceeding from O, and having OA for its axis.

Draw PM perpendicular to OA, and let m represent the ratio of the sine of refraction to the sine of incidence. Then

$$m = OP : Pq;$$

and by similar triangles

$$Pq : PM = OT : OQ = OB^2 : PM \,.\, OQ,$$

since

$$OB^2 = OT \,.\, PM;$$

also

$$OP = OA$$

ultimately. Hence as

$$OA : OB : OQ = a : c : 1,$$

we have ultimately

$$m = \frac{a}{c^2} = \frac{\mu'^2}{\mu} \quad\text{.........................(1)},$$

where μ, μ' denote the ordinary and the principal extraordinary indices of refraction, which are the reciprocals of a, c.

In this particular case the ordinary and extraordinary images cannot be distinguished directly by their polarization, since each consists of rays polarized in all azimuths. But if the objective of the microscope be limited by a narrow aperture, so as to give a predominance to rays lying in one plane, there will in the ordinary image be a predominance of polarization in a plane parallel to the length of the aperture, and in the extraordinary image of polarization in the perpendicular plane.

Next take the case of a uniaxal crystal cut parallel to the axis. In this case, as regards the extraordinary pencil, the divergence after refraction will be different in the axial and equatorial planes, so that a small pencil diverging from a point at the under surface of the crystal will, after refraction, diverge from two focal lines; and in order that a line may be seen distinctly, it must lie in one of the planes of symmetry, in which case, at a certain focal adjustment of the microscope, each element of the line would be seen as a short line in the direction of the actual line, and therefore the line as a whole will be seen sharply defined.

In the equatorial plane the extraordinary ray obeys the ordinary law of refraction; and as regards divergence, therefore, in this plane, on which depends clear vision of a line parallel to the axis, the apparent index will be the same as the real index, μ'. In the axial plane everything will be the same in respect of divergence as in the first case, except that the principal axes of the ellipse which is the section of the extraordinary wave-surface will be interchanged. Accordingly a line in the equatorial plane will be seen distinctly at a focal adjustment which will give an apparent refractive index $\mu^2 : \mu'$.

There will therefore, on the whole, be three focal adjustments of the microscope at which one or other of the systems of cross lines, or both together, will be seen distinctly, namely, one for the extraordinary pencil, which is polarized in the equatorial plane, at which the lines in the axial plane are seen distinctly; another at which the lines in the equatorial plane are seen distinctly; and, intermediate between these, a third for the ordinary pencil, which is polarized in the axial plane, at which both systems at once will be seen distinctly. And the ordinary index, which will be given by the ordinary image, will be a geometric mean between the two apparent extraordinary indices, of which one, namely, that got from the lines in the axial plane, will be the real extraordinary index.

There are two uniaxal crystals, calcite and quartz, for which we know accurately the principal refractive indices for the principal lines of the spectrum from the measures of Rudberg. The principal indices for these two minerals and the apparent indices in the two directions mentioned above are given in the following Table. The indices are given to four places of decimals, and the fixed lines C, D, E are chosen, whence the results applicable to the kinds of light most likely to be employed may be obtained, directly or by interpolation.

Lines	Calcite				Quartz			
	μ	μ'	$\frac{\mu'^2}{\mu}$	$\frac{\mu^2}{\mu'}$	μ	μ'	$\frac{\mu'^2}{\mu}$	$\frac{\mu^2}{\mu'}$
C	1·6545	1·4846	1·3321	1·8438	1·5418	1·5509	1·5601	1·5328
D	1·6585	1·4864	1·3322	1·8505	1·5442	1·5533	1·5624	1·5352
E	1·6636	1·4887	1·3322	1·8590	1·5471	1·5563	1·5656	1·5380

It is well known that the double refraction of quartz differs from that of the generality of uniaxal crystals. Its wave-surface for any colour, instead of being the sphere and spheroid of Huyghens, is a surface of two distinct sheets, which, instead of touching, only make a very close approach along the axis. The polar diameters of the outer, or ordinary, and of the inner, or extraordinary, sheet differ by minute and practically equal quantities from the equatorial diameter of the ordinary sheet. The effect of this, however, on the indices, real or apparent, determined by Mr Sorby's method on a plate cut perpendicular to the axis, would not be sensible. The peculiarity would show itself by giving the two images at different depths circularly polarized, one right-handedly and the other left-handedly.

It may be noticed that the refractive index is given by the reciprocal of the radius of curvature of a section of the wave-surface by a plane perpendicular to the lines seen in focus, and that in order that the lines may be seen distinctly, they must be perpendicular to one of the planes of principal curvature. This rule, as I proceed to show, is general; and it will much simplify the calculation in more complicated cases, by enabling us to dispense with the direct application of Huyghens's construction.

Let O be a point in the first surface of the plate, and consider a small pencil emanating from O in such a direction that its axis, after refraction, is perpendicular to the plate. With centre O describe half a wave-surface, of which only one sheet, DEF, is represented in the figure to avoid confusion. In a direction