MATHEMATICAL AND PHYSICAL PAPERS.

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NOTES ON HYDRODYNAMICS*.

III.—*On the Dynamical Equations.*

IN reducing to calculation the motion of a system of rigid bodies, or of material points. there are two sorts of equations with which we are concerned; the one expressing the geometrical connexions of the bodies or particles with one another, or with curves or surfaces external to the system, the other expressing the relations between the changes of motion which take place in the system and the forces producing such changes. The equations belonging to these two classes may be called respectively the geometrical, and the dynamical equations. Precisely the same remarks apply to the motion of fluids. The geometrical equations which occur in

* The series of "notes on Hydrodynamics" which are printed in Vols. II., III. and IV. of the *Cambridge and Dublin Mathematical Journal*, were written by agreement between Sir William Thomson and myself mainly for the use of Students. As far as my own share in the series is concerned, there is little contained in the "notes" which may not be found elsewhere. Acting however upon the general advice of my friends, I have included my share of the series in the present reprint. It may be convenient to give here the references to the whole series.

I. *On the Equation of Continuity* (Thomson), Vol. II. p. 282.
II. *On the Equation of the Bounding Surface* (Thomson), Vol. III. p. 89.
III. (Stokes) as above.
IV. *Demonstration of a Fundamental Theorem* (Stokes), Vol. III. p. 209.
V. *On the Vis Viva of a Liquid in motion* (Thomson), Vol. IV. p. 90.
VI. *On Waves* (Stokes), Vol. IV. p. 219.

Hydrodynamics have been already considered by Professor Thomson, in Notes I. and II. The object of the present Note is to form the dynamical equations.

The fundamental hypothesis of Hydrostatics is, that the mutual pressure of two contiguous portions of a fluid, separated by an imaginary plane, is normal to the surface of separation. This hypothesis forms in fact the mathematical definition of a fluid. The equality of pressure in all directions is in reality not an independent hypothesis, but a necessary consequence of the former. A proof of this may be seen at the commencement of Prof. Miller's *Hydrostatics*. The truth of our fundamental hypothesis, or at least its extreme nearness to the truth, is fully established by experiment. Some of the nicest processes in Physics depend upon it; for example, the determination of specific gravities, the use of the level, the determination of the zenith by reflection from the surface of mercury.

The same hypothesis is usually made in Hydrodynamics. If it be assumed, the equality of pressure in all directions will follow as a necessary consequence. This may be proved nearly as before, the only difference being that now we have to take into account, along with the impressed forces, forces equal and opposite to the effective forces. The verification of our hypothesis is however much more difficult in the case of motion, partly on account of the mathematical difficulties of the subject, partly because the experiments do not usually admit of great accuracy. Still, theory and experiment have been in certain cases sufficiently compared to shew that our hypothesis may be employed with very little error in many important instances. There are however many phenomena which point out the existence of a tangential force in fluids in motion, analogous in some respects to friction in the case of solids, but differing from it in this respect, that whereas in solids friction is exerted at the surface, and between points which move relatively to each other with a finite velocity, in fluids friction is exerted throughout the mass, where the velocity varies continuously from one point to another. Of course it is the same thing to say that in such cases there is a tangential force along with a normal pressure, as to say that the mutual pressure of two adjacent elements of a fluid is no longer normal to their common surface.

The subsidence of the motion in a cup of tea which has been stirred may be mentioned as a familiar instance of friction, or, which is the same, of a deviation from the law of normal pressure; and the absolute regularity of the surface when it comes to rest, whatever may have been the nature of the previous disturbance, may be considered as a proof that all tangential force vanishes when the motion ceases.

It does not fall in with the object of this Note to enter into the theory of the friction of fluids in motion*, and accordingly the hypothesis of normal pressure will be adopted. The usual notation will be employed, as in the preceding Notes. Consider the elementary parallelepiped of fluid comprised between planes parallel to the coordinate planes and passing through the points whose coordinates are x, y, z, and $x+dx$, $y+dy$, $z+dz$. Let X, Y, Z be the accelerating forces acting on the fluid at the point (x, y, z); then, ρ and X being ultimately constant throughout the element, the moving force parallel to x arising from the accelerating forces which act on the element will be ultimately $\rho X\,dx\,dy\,dz$. The difference between the pressures, referred to a unit of surface, at opposite points of the faces $dy\,dz$ is ultimately $dp/dx\,.\,dx$, acting in the direction of x negative, and therefore the difference of the total pressures on these faces is ultimately $dp/dx\,.\,dx\,dy\,dz$; and the pressures on the other faces act in a direction perpendicular to the axis of x. The effective moving force parallel to x is ultimately $\rho\,.\,D^2x/Dt^2\,.\,dx\,dy\,dz$, where, in order to prevent confusion, D is used to denote differentiation when the independent variables are supposed to be t, and three parameters which distinguish one particle of the fluid from another, as for instance the initial coordinates of the particle, while d is reserved to denote differentiation when the independent variables are x, y, z, t. We have therefore, ultimately,

$$\rho\frac{D^2x}{Dt^2}\,dx\,dy\,dz=\left(\rho X-\frac{dp}{dx}\right)dx\,dy\,dz,$$

* The reader who feels an interest in the subject may consult a memoir by Navier, *Mémoires de l'Académie*, tom. VI. p. 389; another by Poisson, *Journal de l'École Polytechnique*, Cahier XX. p. 139; an abstract of a memoir by M. de Saint-Venant, *Comptes Rendus*, tom. XVII. (Nov. 1843) p. 1240; and a paper in the *Cambridge Philosophical Transactions*, Vol. VIII. p. 287. [*Ante*, Vol. I. p. 75.]

with similar equations for y and z. Dividing by $\rho\, dx\, dy\, dz$, transposing, and taking the limit, we get

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{D^2x}{Dt^2},\quad \frac{1}{\rho}\frac{dp}{dy} = Y - \frac{D^2y}{Dt^2},\quad \frac{1}{\rho}\frac{dp}{dz} = Z - \frac{D^2z}{Dt^2} \ldots\ldots (1).$$

These are the dynamical equations which must be satisfied at every point in the interior of the fluid mass; but they are not at present in a convenient shape, inasmuch as they contain differential coefficients taken on two different suppositions. It will be convenient to express them in terms of differential coefficients taken on the second supposition, that is, that x, y, z, t are the independent variables. Now $Dx/Dt = u$, and on the second supposition u is a function of t, x, y, z, each of which is a function of t on the first supposition. We have, therefore, by Differential Calculus,

$$\frac{Du}{Dt} \text{ or } \frac{D^2x}{Dt^2} = \frac{du}{dt} + \frac{du}{dx}\frac{Dx}{Dt} + \frac{du}{dy}\frac{Dy}{Dt} + \frac{du}{dz}\frac{Dz}{Dt};$$

or, since by the definitions of u, v, w,

$$\frac{Dx}{Dt} = u, \qquad \frac{Dy}{Dt} = v, \qquad \frac{Dz}{Dt} = w,$$

we have
$$\frac{D^2x}{Dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},$$

with similar equations for y and z.

Substituting in (1), we have

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} \\ \frac{1}{\rho}\frac{dp}{dy} &= Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz} \\ \frac{1}{\rho}\frac{dp}{dz} &= Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz} \end{aligned}\right\} \ldots\ldots\ldots\ldots (2),$$

which is the usual form of the equations.

The equations (1) or (2), which are physically considered the same, determine completely, so far as Dynamics alone are concerned, the motion of each particle of the fluid. Hence any other purely dynamical equation which we might set down would be identically satisfied by (1) or (2). Thus, if we were to consider the fluid which at the time t is contained within a closed surface S, and set down the last three equations of equilibrium of a rigid body between the pressures exerted on S, the moving forces due to the accelerating forces acting on the contained fluid, and the effective moving forces reversed, we should not thereby obtain any new equation. The surface S may be either finite or infinitesimal, as, for example, the surface of the elementary parallelepiped with which we started. Thus we should fall into error if we were to set down these three equations for the parallelepiped, and think that we had thereby obtained three new independent equations.

If the fluid considered be homogeneous and incompressible, ρ is a constant. If it be heterogeneous and incompressible, ρ is a function of x, y, z, t, and we have the additional equation $D\rho/Dt = 0$, or

$$\frac{d\rho}{dt} + u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} = 0 \qquad (3),$$

which expresses the fact of the incompressibility. If the fluid be elastic and homogeneous, and at the same temperature θ throughout, and if moreover the change of temperature due to condensation and rarefaction be neglected, we shall have

$$p = k\rho\,(1 + \alpha\theta) \qquad (4),$$

where k is a given constant, depending on the nature of the gas, and α a known constant which is the same for all gases [nearly]. The numerical value of α, as determined by experiment, is ·00366, θ being supposed to refer to the centigrade thermometer.

If the condensations and rarefactions of the fluid be rapid, we may without inconsistency take account of the increase of temperature produced by compression, while we neglect the communication of heat from one part of the mass to another. The only important problem coming under this class is that of sound. If we suppose the changes in pressure and density small, and neglect the squares of small quantities, we have, putting p_1, ρ_1 for the values of p, ρ in equilibrium,

$$\frac{p - p_1}{p_1} = K\frac{\rho - \rho_1}{\rho_1} \qquad (5),$$

K being a constant which, as is well known, expresses the ratio of the specific heat of the gas considered under a constant pressure to its specific heat when the volume is constant. We are not, however, obliged to consider specific heat at all; but we may if we please regard K merely as the value of $d \log p/d \log \rho$ for $\rho = \rho_1$, p being that function of ρ which it is in the case of a mass of air suddenly compressed or dilated. In whichever point of view we regard K, the observation of the velocity of sound forms the best mode of determining its numerical value.

It will be observed that in the proof given of equations (1) it has been supposed that the pressure exerted by the fluid outside the parallelepiped was exerted wholly on the fluid forming the parallelepiped, and not partly on this portion of fluid and partly on the fluid at the other side of the parallelepiped. Now, the pressure arising directly from molecular forces, this imposes a restriction on the diminution of the parallelepiped, namely that its edges shall not become less than the radius of the sphere of activity of the molecular forces. Consequently we cannot, mathematically speaking, suppose the parallelepiped to be indefinitely diminished. It is known, however, that the molecular forces are insensible at sensible distances, so that we may suppose the parallelepiped to become so small that the values of the forces, &c., for any point of it, do not sensibly differ from their values for one of the corners, and that all summations with respect to such elements may be replaced without sensible error by integrations; so that the values of the several unknown quantities obtained from our equations by differentiation, integration, &c. are sensibly correct, so far as this cause of error is concerned; and that is all that we can ever attain to in the mathematical expression of physical laws. The same remarks apply as to the bearing on our reasoning of the supposition of the existence of ultimate molecules, a question into which we are not in the least called upon to enter.

There remains yet to be considered what may be called the dynamical equation of the bounding surface.

Consider, first, the case of a fluid in contact with the surface of a solid, which may be either at rest or in motion. Let P be a point in the surface, about which the curvature is not infinitely great, ω an element of the surface about P, PN a normal at P, directed into the fluid, and let $PN = h$. Through N draw a plane A perpendicular to PN, and project ω on this plane by a circumscribing cylindrical surface. Suppose h greater than the radius r

of the sphere of activity of the molecular forces, and likewise large enough to allow the plane A not to cut the perimeter of ω. For the reason already mentioned r will be neglected, and therefore no restriction imposed on h on the first account. Let Π be the pressure sustained by the solid, referred to a unit of surface, Π having the value belonging to the point P, and let p' be the pressure of the fluid at N. Consider the element of fluid comprised between ω, its projection on the plane A, and the projecting cylindrical surface. The forces acting on this element are, first, the pressure of the fluid on the base, which acts in the direction NP, and is ultimately equal to $p'\omega$; secondly, the pressure of the solid, which ultimately acts along PN and is equal to $\Pi\omega$; thirdly, the pressure of the fluid on the cylindrical surface, which acts everywhere in a direction perpendicular to PN; and, lastly, the moving forces due to the accelerating forces acting on the fluid; and this whole system of forces is in equilibrium with forces equal and opposite to the effective moving forces. Now the moving forces due to the accelerating forces acting on the fluid, and the effective moving forces, are both of the order ωh, and therefore, whatever may be their directions, vanish in the limit compared with the force $p'\omega$, if we suppose, as we may, that h vanishes in the limit. Hence we get from the equation of the forces parallel to PN, passing to the limit,

$$p = \Pi \qquad (6),$$

p being the limiting value of p', or the result obtained by substituting in the general expression for the pressure the coordinates of the point P for x, y, z.

It should be observed that, in proving this equation, the forces on which capillary phenomena depend have not been taken into account. And in fact it is only when such forces are neglected that equation (6) is true.

In the case of a liquid with a free surface, or more generally in the case of two fluids in contact, it may be proved, just as before, that equation (6) holds good at any point in the surface, p, Π being the results obtained on substituting the coordinates of the point considered for the general coordinates in the general expressions for the pressure in the two fluids respectively. In this case, as before, capillary attraction is supposed to be neglected.

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ON THE CONSTITUTION OF THE LUMINIFEROUS ETHER.

THE phenomenon of aberration may be reconciled with the undulatory theory of light, as I have already shown (*Phil. Mag.*, Vol. XXVII. p. 9*), without making the violent supposition that the ether passes freely through the earth in its motion round the sun, but supposing, on the contrary, that the ether close to the surface of the earth is at rest relatively to the earth. This explanation requires us to suppose the motion of the ether to be such, that the expression usually denoted by $udx + vdy + wdz$ is an exact differential. It becomes an interesting question to inquire on what physical properties of the ether this sort of motion can be explained. Is it sufficient to consider the ether as an ordinary fluid, or must we have recourse to some property which does not exist in ordinary fluids, or, to speak more correctly, the existence of which has not been made manifest in such fluids by any phenomenon hitherto observed? I have already attempted to offer an explanation on the latter supposition (*Phil. Mag.*, Vol. XXIX. p. 6†).

In my paper last referred to, I have expressed my belief that the motion for which $udx +$ &c. is an exact differential, which would take place if the ether were like an ordinary fluid, would be unstable; I now propose to prove the same mathematically, though by an indirect method.

Even if we supposed light to arise from vibrations of the ether accompanied by condensations and rarefactions, analogous to the vibrations of the air in the case of sound, since such vibrations would be propagated with about 10,000 times the velocity of the earth,

* *Ante*, Vol. I. p. 134. † *Ante*, Vol. I. p. 153.

we might without sensible error neglect the condensation of the ether in the motion which we are considering. Suppose, then, a sphere to be moving uniformly in a homogeneous incompressible fluid, the motion being such that the square of the velocity may be neglected. There are many obvious phenomena which clearly point out the existence of a tangential force in fluids in motion, analogous in many respects to friction in the case of solids. When this force is taken into account, the equations of motions become (*Cambridge Philosophical Transactions*, Vol. VIII. p. 297*)

$$\frac{dp}{dx} = -\rho\frac{du}{dt} + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \ldots\ldots\ldots\ldots\ldots(1),$$

with similar equations for y and z. In these equations the square of the velocity is omitted, according to the supposition made above, ρ is considered constant, and the fluid is supposed not to be acted on by external forces. We have also the equation of continuity

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the conditions, (1) that the fluid at the surface of the sphere shall be at rest relatively to the surface, (2) that the velocity shall vanish at an infinite distance.

For my present purpose it is not requisite that the equations such as (1) should be known to be true experimentally; if they were even known to be false they would be sufficient, for they may be conceived to be true without mathematical absurdity. My argument is this. If the motion for which $udx + \ldots$ is an exact differential, which would be obtained from the common equations, were stable, the motion which would be obtained from equations (1) would approach indefinitely, as μ vanished, to one for which $udx + \ldots$ was an exact differential, and therefore, for anything proved to the contrary, the latter motion might be stable; but if, on the contrary, the motion obtained from (1) should turn out totally different from one for which $udx + \ldots$ is an exact differential, the latter kind of motion must necessarily be unstable.

Conceive a velocity equal and opposite to that of the sphere impressed both on the sphere and on the fluid. It is easy to prove

* *Ante*, Vol. I. p. 93.

that $udx + \ldots$ will or will not be an exact differential after the velocity is impressed, according as it was or was not such before. The sphere is thus reduced to rest, and the problem becomes one of steady motion. The solution which I am about to give is extracted from some researches in which I am engaged, but which are not at present published. It would occupy far too much room in this Magazine to enter into the mode of obtaining the solution: but this is not necessary; for it will probably be allowed that there is but one solution of the equations in the case proposed, as indeed readily follows from physical considerations, so that it will be sufficient to give the result, which may be verified by differentiation.

Let the centre of the sphere be taken for origin; let the direction of the real motion of the sphere make with the axes angles whose cosines are l, m, n, and let ν be the real velocity of the sphere; so that when the problem is reduced to one of steady motion, the fluid at a distance from the sphere is moving in the opposite direction with a velocity ν. Let a be the sphere's radius: then we have to satisfy the general equations (1) and (2) with the particular conditions

$$u = 0, \quad v = 0, \quad w = 0, \text{ when } r = a \ldots\ldots\ldots\ldots\ldots\ldots(3);$$

$$u = -l\nu, \quad v = -m\nu, \quad w = -n\nu, \text{ when } r = \infty \ldots\ldots\ldots\ldots(4),$$

r being the distance of the point considered from the centre of the sphere. It will be found that all the equations are satisfied by the following values,

$$p = \Pi + \frac{3}{2}\mu\nu\frac{a}{r^3}(lx + my + nz),$$

$$u = \frac{3}{4}\nu\left(\frac{a}{r^3} - \frac{a^3}{r^5}\right)x(lx + my + nz) + l\nu\left(\frac{1}{4}\frac{a^3}{r^3} + \frac{3}{4}\frac{a}{r} - 1\right),$$

with symmetrical expressions for v and w. Π is here an arbitrary constant, which evidently expresses the value of p at an infinite distance. Now the motion defined by the above expressions does not tend, as μ vanishes, to become one for which $udx + \ldots$ is an exact differential, and therefore the motion which would be obtained by supposing $udx + \ldots$ an exact differential, and applying to the ether the common equations of hydrodynamics, would be