MATHEMATICAL AND PHYSICAL PAPERS.

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ON THE STEADY MOTION OF INCOMPRESSIBLE FLUIDS.

[Read April 25, 1842.]

IN this paper I shall consider chiefly the steady motion of fluids in two dimensions. As however in the more general case of motion in three dimensions, as well as in this, the calculation is simplified when $udx + vdy + wdz$ is an exact differential, I shall first consider a class of cases where this is true. I need not explain the notation, except where it may be new, or liable to be mistaken.

To prove that $udx + vdy + wdz$ is an exact differential, in the case of steady motion, when the lines of motion are open curves, and when the fluid in motion has come from an expanse of fluid of indefinite extent, and where, at an indefinite distance, the velocity is indefinitely small, and the pressure indefinitely near to what it would be if there were no motion.

By integrating along a line of motion, it is well known that we get the equation

$$\frac{p}{\rho} = V - \tfrac{1}{2}(u^2 + v^2 + w^2) + C \qquad (1),$$

where $dV = Xdx + Ydy + Zdz$, which I suppose an exact differential. Now from the way in which this equation is obtained, it appears that C need only be constant for the same line of motion, and therefore in general will be a function of the parameter of a line of motion. I shall first shew that in the case considered C is absolutely constant, and then that whenever it is, $udx + vdy + wdz$ is an exact differential*.

To determine the value of C for any particular line of motion, it is sufficient to know the values of p, and of the whole velocity, at any point along that line. Now if there were no motion we should have

$$\frac{p_1}{\rho} = V + C_1 \quad\text{.........................(2),}$$

p_1 being the pressure in that case. But considering a point in this line at an indefinite distance in the expanse, the value of p at that point will be indefinitely nearly equal to p_1, and the velocity will be indefinitely small. Consequently C is more nearly equal to C_1 than any assignable quantity: therefore C is equal to C_1; and this whatever be the line of motion considered; therefore C is constant.

In ordinary cases of steady motion, when the fluid flows in open curves, it does come from such an expanse of fluid. It is conceivable that there should be only a canal of fluid in this expanse in motion, the rest being at rest, in which case the velocity at an infinite distance might not be indefinitely small. But experiment shews that this is not the case, but that the fluid flows in from all sides. Consequently at an indefinite distance the velocity is indefinitely small, and it seems evident that in that case the pressure must be indefinitely near to what it would be if there were no motion.

Differentiating therefore (1) with respect to x, we get

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{dv}{dx} - w\frac{dw}{dx};$$

but $$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz};$$

whence $$v\left(\frac{dv}{dx} - \frac{du}{dy}\right) + w\left(\frac{dw}{dx} - \frac{du}{dz}\right) = 0.$$

[* See note, page 3.]

Similarly, $$w\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+u\left(\frac{du}{dy}-\frac{dv}{dx}\right)=0,$$

$$u\left(\frac{du}{dz}-\frac{dw}{dx}\right)+v\left(\frac{dv}{dz}-\frac{dw}{dy}\right)=0;$$

whence* $$\frac{dv}{dx}=\frac{du}{dy},\quad \frac{dw}{dy}=\frac{dv}{dz},\quad \frac{du}{dz}=\frac{dw}{dx},$$

and therefore $udx+vdy+wdz$ is an exact differential.

When $udx+vdy+wdz$ is an exact differential, equation (1) may be deduced in another way †, from which it appears that C is constant. Consequently, in any case, $udx+vdy+wdz$ is, or is not, an exact differential, according as C is, or is not, constant.

Steady Motion in Two Dimensions.

I shall first consider the more simple case, where $udx+vdy$ is an exact differential. In this case u and v are given by the equations

$$\frac{du}{dx}+\frac{dv}{dy}=0\quad\text{..........................(3)},$$

$$\frac{du}{dy}-\frac{dv}{dx}=0\quad\text{..........................(4)};$$

and p is given by the equation

$$\frac{p}{\rho}=V-\tfrac{1}{2}(u^2+v^2)+C.$$

The differential equation to a line of motion is

$$\frac{dy}{dx}=\frac{v}{u}.$$

* [This conclusion involves an oversight (see *Transactions*, p. 465) since the three preceding equations are not independent, as may readily be seen. I have not thought it necessary to re-write this portion of the paper, since in the two classes of steady motion to which the paper relates, namely those of motion in two dimensions, and of motion symmetrical about an axis, the three analogous equations are reduced to one, and the proposition is true. None of the succeeding results are affected by this error, excepting that the second paragraph of p. 11 must be restricted to the two cases above mentioned.]

† See Poisson, *Traité de Mécanique*.

Now from equation (3) it follows that $udy - vdx$ is always the exact differential of a function of x and y. Putting then

$$dU = udy - vdx,$$

$U = C$ will be the equation to the system of lines of motion, C being the parameter. U may have any value which allows dU/dy and $-dU/dx$ to satisfy the equations which u and v satisfy. The first equation has been already introduced; the second leads to the equation which U is to satisfy; viz.

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0 \quad\text{..........................(5)}.$$

The integral of this equation may be put under different forms. By integrating according to the general method, we get

$$U = F(x + \sqrt{-1}\, y) + f(x - \sqrt{-1}\, y).$$

Now it will be easily seen that U must be wholly real for all values of x and y, at least within certain limits. But $F(\alpha)$ may be put under the form $F_1(\alpha) + \sqrt{-1}\, F_2(\alpha)$, where $F_1(\alpha)$ and $F_2(\alpha)$ are wholly real. Making this substitution in the value of U, we get a result, which, without losing generality, may be put under the form

$$U = F(x + \sqrt{-1}\, y) + F(x - \sqrt{-1}\, y)$$
$$+ \sqrt{-1}\, \{f(x + \sqrt{-1}\, y) - f(x - \sqrt{-1}\, y)\},$$

changing the functions.

If we develope these functions in series ascending according to integral powers of y, by Taylor's Theorem, which can always be done as long as the origin is arbitrary, we get a series which I shall write for shortness,

$$U = 2\cos\left(\frac{d}{dx}y\right)F(x) - 2\sin\left(\frac{d}{dx}y\right)f(x),$$

the same result as if we had integrated at once by series by Maclaurin's Theorem.

It has been proved that the general integral of (5) may be put under the form

$$U = \Sigma A^{\alpha x + \beta y},$$

where $\alpha^2+\beta^2=0$. Consequently α and β must be, one real, the other imaginary, or both partly real and partly imaginary. Putting then $\alpha=\alpha_1+\sqrt{-1}\,\alpha_2$, $\beta=\beta_1+\sqrt{-1}\,\beta_2$, introducing the condition that $\alpha^2+\beta^2=0$, and replacing imaginary exponentials by sines and cosines, we find that the most general value of U is of the form

$$U=\Sigma A e^{n(\cos\gamma\,.\,x-\sin\gamma\,.\,y+a)}.\cos n\,(\sin\gamma\,.\,x+\cos\gamma\,.\,y+b),$$

where A, n, γ, a and b have any real values, the value of U being supposed to be real.

If we take the value of U,

$$U=2\cos\left(\frac{d}{dx}\,y\right)F(x)-2\sin\left(\frac{d}{dx}\,y\right)f(x),$$

and develope each term, such as ax^n, in $F(x)$ or $f(x)$, in a series, and then sum the series by the formula

$$\cos n\theta+\sqrt{-1}\sin n\theta=\cos^n\theta\left(1+\frac{n}{1}\sqrt{-1}\tan\theta-\ldots\right),$$

we find that the general value of U takes the form

$$U=\Sigma A r^n\cos(n\theta+B).$$

As long as the origin of x is arbitrary, only integral powers of x will enter into the development $F(x)$ and $f(x)$, and therefore the above series will contain only integral values of n. For particular positions of the origin however, fractional powers may enter. The equation

$$\frac{d^2U}{dr^2}+\frac{1}{r}\,\frac{dU}{dr}+\frac{1}{r^2}\,\frac{d^2U}{d\theta^2}=0,$$

which (5) becomes when transferred to polar co-ordinates, is satisfied by the above value of U, whatever n be, even if it be imaginary, in which case the value of U takes the form

$$U=\Sigma A r^m e^{n\theta}\cos(m\theta-\log_e r^n+B).$$

We may employ equation (5), to determine whether a proposed system of lines can be a system in which fluid can move, the motion being of the kind for which $udx+vdy$ is an exact differential.

Let $f(x,y)=U_1=C$ be the equation to the system, C being the parameter. Then, if the motion be possible, some value of

U which satisfies (5) must be constant for all values of x and y for which U_1 is constant. Consequently this value must be a function of U_1. Let it $= \phi(U_1)$. Then, substituting this value in (5), and performing the differentiations, we get

$$\phi''(U_1)\left\{\left(\frac{dU_1}{dx}\right)^2 + \left(\frac{dU_1}{dy}\right)^2\right\} + \phi'(U_1)\left\{\frac{d^2U_1}{dx^2} + \frac{d^2U_1}{dy^2}\right\} = 0,$$

or

$$\frac{\phi''(U_1)}{\phi'(U_1)} + \frac{\dfrac{d^2U_1}{dx^2} + \dfrac{d^2U_1}{dy^2}}{\left(\dfrac{dU_1}{dx}\right)^2 + \left(\dfrac{dU_1}{dy}\right)^2} = 0 \ldots\ldots\ldots\ldots\ldots(6).$$

Now, if the motion be possible, the second term of this equation must be a function of U_1; x, y and U_1 being connected by the equation $f(x, y) = U_1$. Consequently, if by means of this latter equation we eliminate x or y from the second term of (6), the other must disappear. If it does not, the motion is impossible; if it does, the integration of equation (6), in which the variables are separated, will give $\phi(U_1)$ under the form

$$\phi(U_1) = AF(U_1) + B,$$

A and B being the arbitrary constants. The values of u and v will immediately be got by differentiation, and then p will be known. Nothing will be left arbitrary but a constant multiplying the values of u and v, and another added to the value of p.

I shall mention a few examples. Let $U = ar^{\frac{1}{2}} \cos \frac{1}{2}\theta$. In this case the lines of motion are similar parabolas about the same focus. The velocity at any point varies inversely as the square root of the distance from the focus.

Again, let $U = axy$. In this case the lines of motion are rectangular hyperbolas about the same asymptotes. Also,

$$u = \frac{dU}{dy} = ax, \text{ and } v = -\frac{dU}{dx} = -ay.$$

In this case therefore the velocity varies as the distance from the centre, and the particles in a section parallel to either of the axes remain in a section parallel to that axis.

I shall now consider the general case, where $udx + vdy$ need not be an exact differential.

In this case p, u and v, are given by the equations

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy} \quad\text{.................(7)},$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - u\frac{dv}{dx} - v\frac{dv}{dy} \quad\text{.................(8)},$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad\text{........................(9)}.$$

We still have $\frac{dy}{dx} = \frac{v}{u}$, for the differential equation to a line of motion, where $udy - vdx$ is still an exact differential, on account of equation (9). Eliminating p by differentiation from (7) and (8), and expressing the result in terms of U, we get the equation which U is to satisfy, viz.

$$\frac{dU}{dy}\frac{d}{dx}\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) - \frac{dU}{dx}\frac{d}{dy}\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) = 0,$$

or, for shortness,

$$\left(\frac{dU}{dy}\frac{d}{dx} - \frac{dU}{dx}\frac{d}{dy}\right)\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right) = 0 \quad\text{.........(10)*}.$$

* [This equation may be applied to prove an elegant theorem due to Mr F. D. Thomson {see the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, Vol. III. (1866), p. 238, and Vol. IV. p. 37}, that if a vessel bounded by a cylindrical surface of any kind and by two planes perpendicular to its generating lines be filled with homogeneous liquid, and the whole be revolving uniformly about a fixed axis parallel to its generating lines, then if the vessel be suddenly arrested the motion of the liquid will be steady.

If ω be the angular velocity, we shall have for the motion before impact

$$U = -\int(\omega y\,dy + \omega x\,dx) = -\tfrac{1}{2}\omega(x^2 + y^2) = -\tfrac{1}{2}\omega r^2,$$

omitting the constant as unnecessary. If u, v be the components of the change of velocity produced by impact, it follows from the equations of impulsive motion that $udx + vdy$ will be a perfect differential $d\phi$, where ϕ satisfies the partial differential equation $\nabla\phi = 0$, ∇ standing for $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$. If U' be the U-function corresponding to this motion—and such a function exists by virtue of the equation of continuity whether the motion be steady or not—we have

$$U' = \int\left(\frac{d\phi}{dx}dy - \frac{d\phi}{dy}dx\right) = \int\left(\frac{d\phi}{dr}r\,d\theta - \frac{1}{r}\frac{d\phi}{d\theta}dr\right),$$

where the quantity under the sign $\int$ is a perfect differential by virtue of the equa-

In this case, since $p = \int \left(\frac{dp}{dx} dx + \frac{dp}{dy} dy \right)$, equations (7) and (8) give

tion $\nabla\phi = 0$; and we see at once that $\nabla U' = 0$. Hence for the whole motion just after impact

$$\nabla (U + U') = \nabla U = -2\omega,$$

which satisfies the equation of steady motion (10); and as the condition at the boundary, namely that the fluid shall slide along it, is satisfied, being satisfied initially, it follows that the initial motion after impact will be continued as steady motion.

To actually determine the function ϕ or U', and thereby the motion in any given case, we must satisfy not only the general equation $\nabla\phi = 0$ but also the equation of condition at the boundary, namely that there shall be no velocity in a direction normal to the surface, which gives

$$\left(\frac{d\phi}{dx} - \omega y \right) dy - \left(\frac{d\phi}{dy} - \omega x \right) dx = 0 \quad \ldots\ldots (\alpha),$$

at any point of the boundary. If $f(x, y) = 0$ be the equation of the boundary, we must substitute $-df/dx \div df/dy$ for dy/dx in (α), and the resulting equation will have to be satisfied when $f = 0$ is satisfied.

There are but few forms of boundary for which the solution of the problem can be actually effected analytically, among which may be mentioned in particular the case of a rectangle. But by taking particular solutions of the equation $\nabla\phi = 0$, substituting in (α) and integrating, which gives

$$-\tfrac{1}{2}\omega r^2 + U' = C \quad \ldots\ldots (\beta),$$

or what comes to the same thing taking particular solutions of the equation $\nabla U' = 0$ and substituting in (β), which gives the general equation of the lines of motion, we may synthetically obtain an infinity of examples in which the conditions of the problem are satisfied, any one of the lines of motion being taken as the boundary of the fluid.

Thus for $U' = kr^3 \cos 3\theta$ we have for the lines of motion

$$-\tfrac{1}{2}\omega r^2 + kr^3 \cos 3\theta = C \quad \ldots\ldots (\gamma),$$

or

$$-\tfrac{1}{2}\omega r^2 + k\{4(r\cos\theta)^3 - 3r^2 . r\cos\theta\} = C \quad \ldots\ldots (\delta),$$

which therefore are cubic curves, recurring when θ is increased by 120°. (δ) is satisfied by

$$r\cos\theta = a,$$

giving a straight line, provided

$$k = -\frac{\omega}{6a}, \qquad C = 4ka^3 = -\tfrac{2}{3}\omega a^2.$$

Hence when k has the above value the cubic curve (γ) breaks up, for the particular value of the parameter C above written, into three straight lines forming the sides of an equilateral triangle, and the vessel may therefore be supposed to be an equilateral triangular prism. The various lines of motion correspond to values of the parameter C from 0 to $-\tfrac{2}{3}\omega a^2$. This case is given by Mr Thomson.

$U' = kr^2 \cos 2\theta$ leads to the case of steady motion in similar and concentric ellipses considered in the text a little further on, which therefore may be conceived to have been produced from motion about a fixed axis as pointed out by Mr Thomson. In fact, any case of steady motion in two dimensions in which $\nabla U = \text{const.}$ may be conceived to have been so produced.]

$$\frac{p}{\rho} = V - \int\left\{\left(\frac{dU}{dy}\frac{d^2U}{dxdy} - \frac{dU}{dx}\frac{d^2U}{dy^2}\right)dx + \left(\frac{dU}{dx}\frac{d^2U}{dxdy} - \frac{dU}{dy}\frac{d^2U}{dx^2}\right)dy\right\}.$$

Now $\frac{1}{2}d\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\} = \left(\frac{dU}{dx}\frac{d^2U}{dx^2} + \frac{dU}{dy}\frac{d^2U}{dxdy}\right)dx + \left(\frac{dU}{dx}\frac{d^2U}{dxdy} + \frac{dU}{dy}\frac{d^2U}{dy^2}\right)dy$;

whence,

$$\frac{dU}{dy}\frac{d^2U}{dxdy}dx + \frac{dU}{dx}\frac{d^2U}{dxdy}dy - \frac{dU}{dx}\frac{d^2U}{dy^2}dx - \frac{dU}{dy}\frac{d^2U}{dx^2}dy$$
$$= \frac{1}{2}d\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\} - \left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right);$$

and therefore

$$\frac{p}{\rho} = V - \frac{1}{2}\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\} + \int\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right),$$
$$= V - \frac{1}{2}(v^2 + u^2) + \int\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}\right)dU.$$

It will be observed that $\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = \chi(U)$, is a first integral of (10). Consequently this latter term, which is the value of C in (1), comes out a function of the parameter of a line of motion as it should.

We may employ equation (10), precisely as before, to enquire whether a proposed system of lines can, under any circumstances, be a system of lines of motion. Let $f(x, y) = U_1 = C$, be the equation to the system; then, putting as before, $U = \phi(U_1)$, we get

$$\phi''(U_1)\left(\frac{dU_1}{dy}\frac{d}{dx} - \frac{dU_1}{dx}\frac{d}{dy}\right)\left\{\left(\frac{dU_1}{dx}\right)^2 + \left(\frac{dU_1}{dy}\right)^2\right\}$$
$$+ \phi'(U_1)\left(\frac{dU_1}{dy}\frac{d}{dx} - \frac{dU_1}{dx}\frac{d}{dy}\right)\left(\frac{d^2U_1}{dx^2} + \frac{d^2U_1}{dy^2}\right) = 0;$$

or, $$P\phi''(U_1) + Q\phi'(U_1) = 0, \text{ suppose.}$$

Hence, as before, if we express y in terms of x and U_1, from the equation $f(x, y) = U_1$, and substitute that value in $\frac{Q}{P}$, the result must not contain x. If it does, the proposed system of lines cannot be a system of lines of motion; if not, the integration of the above equation will give $\phi(U_1)$, under the form

$$\phi(U_1) = AF(U_1) + B,$$

and we can immediately get the values of u, v and p, with the same arbitrary constants as in the previous case.

One case in which the motion is possible is where the lines of motion are a system of similar ellipses or hyperbolas about the same centre, or a system of equal parabolas having the same axis. In the case of the ellipse, the particles in a radius vector at any time remain in a radius vector, and the value of p has the form

$$\rho V + A + B(x^2 + y^2).$$

When however the ellipse becomes a circle, P and Q vanish in the equation $P\phi''(U_1) + Q\phi'(U_1) = 0$. Consequently the form of ϕ may be any whatever. The value of U_1 being $x^2 + y^2$, we have

$$u = 2\phi'(U_1)\,y, \quad v = -2\phi'(U_1)\,x;$$

whence, $$u^2 + v^2 = 4\{\phi'(U_1)\}^2(x^2 + y^2) = 4U_1\{\phi'(U_1)\}^2.$$

Hence, the velocity may be any function of the distance from the centre. It is evident that we may conceive cylindrical shells of fluid, having a common axis, to be revolving about that axis with any velocities whatever, if we do not consider friction, or whether such a mode of motion would be stable. The result is the same if we enquire in what way fluid can move in a system of parallel lines.

In any case where the motion in a certain system of lines is possible, if we suppose two of these lines to be the bases of bounding cylindrical surfaces, and if we suppose the velocity and direction of motion, at each point of a section of the entering, and also of the issuing fluid, to be what that case requires, I have not proved that the fluid *must* move in that system of lines. When the above conditions are given there may still perhaps be different modes of steady motion; and of these some may be stable, and others unstable. There may even be no stable steady mode of