Chapter I

PRELIMINARY MATHEMATICS

1·1. We propose in this chapter to give a brief account of some mathematical ideas with which the reader must be familiar in order to be able to understand what follows in this volume.

1·2. Surface and volume integrals. Though the process of evaluating surface and volume integrals in general involves double or triple integration and must be learnt from books on Analysis, yet in theoretical work in Applied Mathematics considerable use is made of surface and volume integrals without evaluation, and we propose here merely to explain what is implied when such symbols as

$$\int f(x, y, z)\, dS \quad \text{and} \quad \int f(x, y, z)\, dv$$

are used to denote integration over a surface and throughout a volume.

A definite integral of a function of one variable, say $\int_a^b f(x)\, dx$, may be defined thus: let the interval from a to b on the x-axis be divided into any number of sub-intervals $\delta_1, \delta_2, \ldots \delta_n$, and let f_r denote the value of $f(x)$ at some point on δ_r; let the sum $\sum_{r=1}^{n} f_r \delta_r$ be formed and let the number n be increased without limit. Then, provided that the limit as $n \to \infty$ of $\sum_{r=1}^{n} f_r \delta_r$ exists and is independent of the method of division into sub-intervals and of the choice of the point on δ_r at which the value of $f(x)$ is taken, this limit is the definite integral of $f(x)$ from a to b.

In the same way we may define $\int f(x, y, z)\, dS$ over a given surface; let the given surface be divided into any number of small parts $\delta_1, \delta_2, \ldots \delta_n$ and let f_r denote the value of $f(x, y, z)$

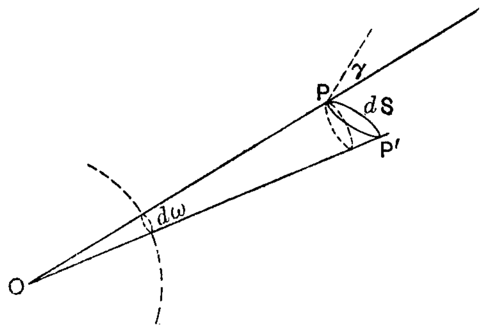
at some point on δ_r, then the limit as $n \to \infty$ of $\sum_{r=1}^{n} f_r \delta_r$, provided the limit exists under the same conditions as aforesaid, is defined to be the integral $\int f(x, y, z)\, dS$ over the given surface.

Any difficulty as to the precise meaning to be attributed to 'area of a curved surface' may be avoided thus: after choosing the point on each sub-division δ_r of the surface at which the value of $f(x, y, z)$ is taken, project this element of surface on to the tangent plane at the chosen point, and take the plane projection of the element as the measure of δ_r in forming the sum.

The integral $\int f(x, y, z)\, dv$ through a given volume may be defined in the same way.

1·3. Solid angles. The solid angle of a cone *is measured by the area intercepted by the cone on the surface of a sphere of unit radius having its centre at the vertex of the cone.*

The solid angle subtended at a point by a surface of any form is measured by the solid angle of the cone whose vertex is at the given point and whose base is the given surface.



Let PP' be a small element of area dS which subtends a solid angle $d\omega$ at O.

Let the normal to dS make an acute angle γ with OP and let $OP = r$. Then the cross-section at P of the cone which PP' subtends at O is $dS \cos\gamma$, and this cross-section and the small area $d\omega$ intercepted on the unit sphere are similar figures, so that

$$dS\cos\gamma : d\omega = r^2 : 1.$$

Whence

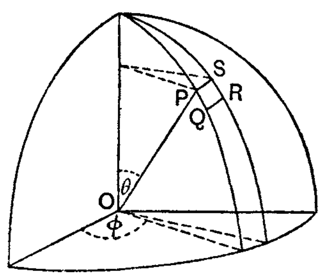
$$\left.\begin{aligned} d\omega &= (dS\cos\gamma)/r^2 \\ \text{or}\quad dS &= r^2\sec\gamma\, d\omega \end{aligned}\right\} \qquad \text{..................(1).}$$

It follows that the area of a finite surface can be represented as an integral over a spherical surface, thus

$$S = \int r^2 \sec\gamma\, d\omega \qquad \text{....................(2)}$$

with suitable limits of integration.

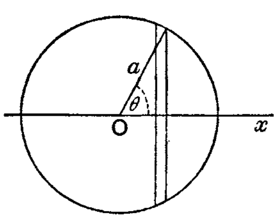
1·31. $d\omega$ in polar co-ordinates. $d\omega$ is an element of the surface of a unit sphere. Let the element be $PQRS$ bounded by meridians and small circles, where the angular co-ordinates of P are θ, ϕ. Then since the arc PS subtends an angle $d\phi$ at the centre of a circle of radius $\sin\theta$, therefore $PS = \sin\theta\, d\phi$; and $PQ = d\theta$, so that

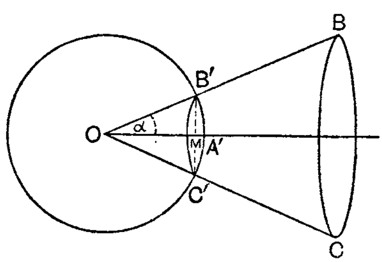


$$d\omega = PQ\,.\,PS = \sin\theta\, d\theta\, d\phi.$$

1·32. Solid angle of a right circular cone. A narrow zone of a sphere of radius a cut off between parallel planes may be regarded as a circular band of breadth $a\,d\theta$ and radius $a\sin\theta$, so that its area $= 2\pi a^2 \sin\theta\, d\theta$

$$= -2\pi a\, dx, \text{ where } x = a\cos\theta.$$





Hence the area of a zone of finite breadth

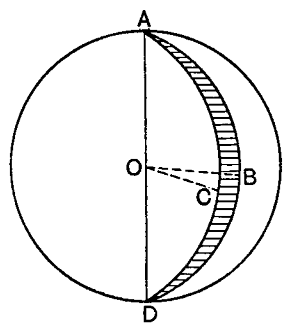
$$= 2\pi a\,(x_1 - x_2)$$
$$= \text{circumference of sphere} \times \text{axial breadth of zone.}$$

A right circular cone BOC of vertical angle 2α intercepts on a unit sphere of centre O a cap $B'A'C'$ of height $MA' = 1 - \cos\alpha$ and area

$$2\pi(1-\cos\alpha) \quad \text{.....................(1)},$$

so that this is the measure of the solid angle of the cone.

1·33. The idea of the solid angle may easily be extended, if we observe that any bounded area on a unit sphere may be regarded as measuring a solid angle. Thus a lune bounded by semi-circles ABD, ACD may be taken as measuring the solid angle between the diametral planes ABD, ACD.



Let α be the angle between these planes. Because of the symmetry about AD, it is evident that

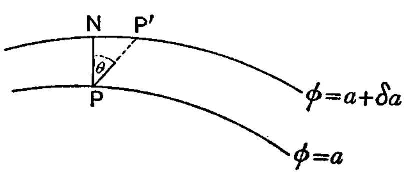
area of lune : area of sphere $= \alpha : 2\pi$.

But the area of the sphere is 4π, so that the area of the lune is 2α; or the solid angle between two planes is twice their inclination to one another.

1·4. Scalar functions of position and their gradients. Let $\phi(x, y, z)$ be a continuous single-valued function of the position of a point in some region of space. Suppose that the function ϕ is not constant throughout any region, so that the equation

$$\phi(x, y, z) = \text{const.}$$

represents a surface. We assume that through each point of the region in which ϕ is defined, there passes a surface $\phi = \text{const.}$ We also assume that at every point P on this surface there is a definite normal PN and that the tangent plane at P varies continuously with the position of P on the surface.



From the definition of ϕ two surfaces

$$\phi(x, y, z) = a \quad \text{and} \quad \phi(x, y, z) = b$$

cannot intersect; for if they had a common point it would be a point at which ϕ had more than one value, in contradiction to the hypothesis that ϕ is a single-valued function.

Consider two neighbouring surfaces

$$\phi = a \quad \text{and} \quad \phi = a + \delta a.$$

Let P, P' be points on each and let the normal at P to $\phi = a$ meet $\phi = a + \delta a$ in N. For small values of δa PN will also be normal to $\phi = a + \delta a$.

Then using ϕ_P to denote the value of ϕ at P, we have

$$\frac{\phi_{P'} - \phi_P}{PP'} = \frac{\delta a}{PP'} = \frac{\phi_N - \phi_P}{PP'} = \frac{\phi_N - \phi_P}{PN} \cdot \frac{PN}{PP'}$$

$$= \frac{\phi_N - \phi_P}{PN} \cos\theta,$$

where θ is the angle NPP'.

Now if $PP' = \delta s$ and $PN = \delta n$, and we make δa and therefore δs and δn tend to zero, the limit of $(\phi_{P'} - \phi_P)/PP'$ is the rate of increase of ϕ in the direction δs and is denoted by $\frac{\partial \phi}{\partial s}$; and similarly the limit of $(\phi_N - \phi_P)/PN$ is the rate of increase of ϕ in the normal direction δn and is denoted by $\frac{\partial \phi}{\partial n}$, and we have

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial n} \cos\theta \quad \text{.........................(1).}$$

Thus we have proved that the space rate of increase of ϕ in any direction δs is the component in that direction of its space rate of increase in the direction normal to the surface $\phi = $ const.; or that if we construct a vector of magnitude $\partial\phi/\partial n$ in direction PN, then the component of this vector in any direction is the space rate of increase of ϕ in that direction.

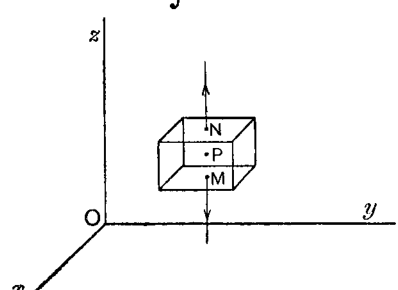
The vector $\partial\phi/\partial n$ with its proper direction is called the **gradient of** ϕ and written **grad** ϕ.

To recapitulate: ϕ is a continuous *scalar* function of position having a definite single value at each point of a certain region of space, and the *gradient of* ϕ is defined in this way: through any point P in the region there passes a surface $\phi = $ const., then *a vector* normal to this surface at P whose magnitude is the space rate of increase of ϕ in this normal direction is defined to be the *gradient of* ϕ at P, and it has the property that its component in any direction gives the space rate of increase of ϕ at P in that direction. It is clear that the gradient measures the greatest rate of increase of ϕ at a point.

1·5. A vector field. If to every point of a given region there corresponds a definite vector **A**, generally varying its magnitude and direction from point to point, then the region is called a **vector field**, or the field of the vector **A**; e.g. electric field, magnetic field.

1·51. Flux of a vector. If a surface S be drawn in the field of a vector **A** and A_n denotes the component of **A** normal to an element dS of the surface, then the integral $\int A_n dS$ is called the *flux of A through S*. Since a surface has two sides the sense of the normal must be taken into account, and the sign of the flux is changed when the sense of the normal is changed. The flux of a vector through a surface is clearly a scalar magnitude.

1·52. Divergence of a vector field. Let **A** denote a vector field which has no discontinuities throughout a given region of space. Let δv denote any small element of volume containing a point P in the region and let $\int A_n dS$ denote the outward flux of **A** through the boundary of δv, then the limit as $\delta v \to 0$ of $\dfrac{\int A_n dS}{\delta v}$ is defined to be the **divergence** of **A** at the point P and denoted by div **A**.



It can be shewn that, subject to certain conditions, this limit is independent of the shape of the element of volume δv, but for our present purpose, which is to obtain a Cartesian form for div **A**, it will suffice to calculate the limit for a rectangular element of volume. Using rectangular axes let P be the centre (x, y, z)

of a small rectangular parallelepiped with edges parallel to the axes of lengths δx, δy, δz.

Let the vector $\mathbf{A}$ have components A_x, A_y, A_z parallel to the axes at P.

Consider the contributions of the faces of the parallelepiped to the flux of the vector out of the element of volume. The two faces parallel to the xy plane are of area $\delta x\,\delta y$, the component of $\mathbf{A}$ normal to these at the centre (x, y, z) of the parallelepiped is A_z. The co-ordinates of the centres M, N of these faces are $x, y, z-\frac{1}{2}\delta z$ and $x, y, z+\frac{1}{2}\delta z$; so that if the magnitude of A_z at P is $f(x, y, z)$, its magnitude at M is $f(x, y, z-\frac{1}{2}\delta z)$ or $f(x, y, z)-\frac{1}{2}\frac{\partial f}{\partial z}\delta z$, to the first power of δz, i.e. $A_z-\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z$, and similarly the magnitude at N is $A_z+\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z$, and both these components are in the direction Oz. Then assuming what can easily be proved, that, subject to certain conditions, the average value of the component over each small rectangle is the value at its centre, the contributions of these two faces to the total *outward* flux are

$$-\left(A_z-\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z\right)\delta x\,\delta y \quad \text{and} \quad \left(A_z+\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z\right)\delta x\,\delta y,$$

giving a sum

$$\frac{\partial A_z}{\partial z}\delta x\,\delta y\,\delta z.$$

Finding similarly the contributions of the other two pairs of faces, we have for the total outward flux

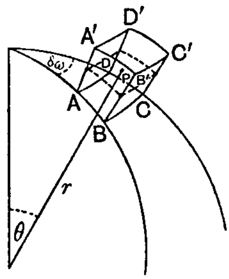
$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\delta x\,\delta y\,\delta z$$

to this order of small quantities.

But the volume δv of the small element is $\delta x\,\delta y\,\delta z$, so that in accordance with our definition, dividing the flux by the volume and proceeding to the limit in which the terms of higher order in the numerator disappear, we have

$$\operatorname{div}\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \quad \text{...............(1).}$$

1·53. Divergence in polar co-ordinates. Let P be the point (r, θ, ω) and suppose it to be the centre of an element of volume $ABCDA'B'C'D'$, whose faces $ABCD$, $A'B'C'D'$ are portions of spheres of radii $r \mp \frac{1}{2}\delta r$, $ADD'A'$, $BCC'B'$ are portions of cones of angles $\theta \mp \frac{1}{2}\delta\theta$, and $ABB'A'$, $DCC'D'$ are planes $\omega \mp \frac{1}{2}\delta\omega$. The lengths of the edges of the element of volume are δr, $r\delta\theta$ and $r \sin\theta\,\delta\omega$ and its volume is $r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\omega$.



Let A_r, A_θ, A_ω denote the components of the vector $\mathbf{A}$ at P in the directions in which r, θ, ω increase, i.e. perpendicular to the faces of the element. The cross-section of the element through P at right angles to A_r is of area $r^2 \sin\theta\,\delta\theta\,\delta\omega$, so that the flux of $\mathbf{A}$ through this cross-section is $A_r r^2 \sin\theta\,\delta\theta\,\delta\omega$. Hence the outward flux across the parallel section $ABCD$ which only differs from that through P by having $r - \frac{1}{2}\delta r$ instead of r is

$$-\left\{A_r r^2 \sin\theta\,\delta\theta\,\delta\omega - \tfrac{1}{2}\delta r\,.\frac{\partial}{\partial r}(A_r r^2 \sin\theta\,\delta\theta\,\delta\omega)\right\} + \epsilon_1,$$

and the flux across $A'B'C'D'$ is in like manner

$$+\left\{A_r r^2 \sin\theta\,\delta\theta\,\delta\omega + \tfrac{1}{2}\delta r\,.\frac{\partial}{\partial r}(A_r r^2 \sin\theta\,\delta\theta\,\delta\omega)\right\} + \epsilon_2,$$

where ϵ_1, ϵ_2 are small quantities of the fourth order in δr, $\delta\theta$, $\delta\omega$.

Hence this pair of opposite faces contribute an amount

$$\frac{\partial}{\partial r}(r^2 A_r) \sin\theta\,\delta r\,\delta\theta\,\delta\omega + \epsilon_1 + \epsilon_2$$

to the total outward flux.

It may be shewn in the same way that the faces $ADD'A'$ and $BCC'B'$ contribute

$$-\left\{A_\theta r \sin\theta\,\delta r\,\delta\omega - \tfrac{1}{2}\delta\theta\,.\frac{\partial}{\partial \theta}(A_\theta r \sin\theta\,\delta r\,\delta\omega)\right\} + \epsilon_3$$

and $$+\left\{A_\theta r \sin\theta\,\delta r\,\delta\omega + \tfrac{1}{2}\delta\theta\,.\frac{\partial}{\partial \theta}(A_\theta r \sin\theta\,\delta r\,\delta\omega)\right\} + \epsilon_4;$$

and that the faces $ABB'A'$, $DCC'D'$ contribute

$$-\left\{A_\omega r\,\delta\theta\,\delta r - \tfrac{1}{2}\delta\omega\,.\,\frac{\partial}{\partial\omega}(A_\omega r\,\delta\theta\,\delta r)\right\} + \epsilon_5$$

and
$$+\left\{A_\omega r\,\delta\theta\,\delta r + \tfrac{1}{2}\delta\omega\,.\,\frac{\partial}{\partial\omega}(A_\omega r\,\delta\theta\,\delta r)\right\} + \epsilon_6,$$

where ϵ_3, ϵ_4, ϵ_5, ϵ_6 have like meanings.

Hence the total outward flux from the six faces is

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\omega}{\partial\omega}\right\} r^2\sin\theta\,\delta r\,\delta\theta\,\delta\omega + \epsilon,$$

where ϵ is of the fourth order in δr, $\delta\theta$, $\delta\omega$.

Therefore if we divide the flux by the volume and then make δr, $\delta\theta$, $\delta\omega$ tend to zero, we get for the divergence at P

$$\text{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\omega}{\partial\omega} \quad \ldots(1).$$

1·54. Divergence in cylindrical co-ordinates. Using cylindrical co-ordinates r, θ, z and taking an element of volume of edges δr, $r\,\delta\theta$, δz with its centre at (r, θ, z), it can be shewn in the same way that

$$\text{div}\,\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z},$$

where A_r, A_θ, A_z are the components of $\mathbf{A}$ in the directions of the increments in r, θ and z.

Chapter II

INTRODUCTION TO ELECTROSTATICS

2·1. The electric field. It was known to the Greeks and Romans that when pieces of amber are rubbed they acquire the power of attracting to themselves light bodies. There are other substances which possess the same property; thus, if a stick of sealing-wax is rubbed on a piece of dry cloth it will attract bran or small scraps of paper sufficiently near to it. The same result is obtained if a glass rod is rubbed with a dry piece of silk. It is also found that the cloth and the silk acquire the same property as the sealing-wax and the glass rod. Further, if the sealing-wax is suspended so that it is free to move, it is found that the cloth attracts the sealing-wax, but two pieces of sealing-wax similarly treated repel one another. In the same way the glass rod and the silk attract one another, but two such glass rods repel one another.

We describe bodies in such a state as *electrified* or *charged with electricity*. The word was derived by William Gilbert* from ἤλεκτρον or *amber*, the first substance upon which such experiments were performed.

If we experiment further we find that an electrified stick of sealing-wax is attracted by an electrified glass rod, but repelled by the piece of silk with which the rod has been rubbed.

These and kindred phenomena are explained by the statement that electricity is of two kinds, and that charges of the same kind repel while charges of opposite kinds attract one another.

It is convenient to describe the two kinds of electricity as *positive* and *negative*, that produced as above on the sealing-wax is called negative and that on the glass rod positive; but it must be pointed out that this is merely a convenient arbitrary nomenclature and that the opposite would have answered all purposes equally well.

* William Gilbert (1540–1603), a native of Colchester, Fellow of St John's College, Cambridge.