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$$C = a + 2^5 b + 3^5 c + 4^5 d + \&c.,$$
$$D = a + 2^7 b + 3^7 c + 4^7 d + \&c.,$$
$$\&c., \qquad \&c.$$

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$$\frac{\pi}{2}\phi(x) = \sin x \int_0^\pi da\, \phi(a) \sin a + \sin 2x \int_0^\pi da\, \phi(a) \sin 2a + \sin 3x \int_0^\pi da\, \phi(a) \sin 3a + \&c.,$$

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Each of the coefficients is a definite integral. We have in general

$$2\pi A = \int_{-\pi}^{+\pi} dx\, F(x), \qquad \pi a_i = \int_{-\pi}^{+\pi} dx F(x) \cos ix,$$

and

$$\pi b_i = \int_{-\pi}^{+\pi} dx\, F(x) \sin ix.$$

We thus form the general theorem, which is one of the chief elements of our analysis:

$$2\pi F(x) = \Sigma_{i=-\infty}^{i=+\infty} \left(\cos ix \int_{-\pi}^{+\pi} d\alpha F(\alpha) \cos i\alpha + \sin ix \int_{-\pi}^{+\pi} d\alpha F(\alpha) \sin i\alpha\right),$$

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363. The foregoing solution shews also the variable movement of heat in an infinite line, one point of which is submitted to a constant temperature . 352

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OF THE FREE MOVEMENT OF HEAT IN AN INFINITE SOLID.

372—376. The expression for the variable movement of heat in an infinite solid mass, according to three dimensions, is derived immediately from that of the linear movement. The integral of the equation

$$\frac{dv}{dt}=\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}$$

solves the proposed problem. It cannot have a more extended integral; it is derived also from the particular value

$$v=e^{-n^2t}\cos nx,$$

or from this:

$$v=\frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}},$$

which both satisfy the equation $\frac{dv}{dt}=\frac{d^2v}{dx^2}$. The generality of the integrals obtained is founded upon the following proposition, which may be regarded as self-evident. Two functions of the variables x, y, z, t are necessarily identical, if they satisfy the differential equation

$$\frac{dv}{dt}=\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2},$$

and if at the same time they have the same value for a certain value of t 368

377—382. The heat contained in a part of an infinite prism, all the other points of which have nul initial temperature, begins to be distributed throughout the whole mass; and after a certain interval of time, the state of any part of the solid depends not upon the distribution of the initial heat, but simply upon its quantity. The last result is not due to the increase of the distance included between any point of the mass and the part which has been heated; it is entirely due to the increase of the time elapsed. In all problems submitted to analysis, the exponents are absolute numbers, and not quantities. We ought not to omit the parts of these exponents which are incomparably smaller than the others, but only those whose absolute values are extremely small . . 376

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$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}+2\frac{d^4v}{dx^2.dy^2}+\frac{d^4v}{dy^4}=0......(e),$$

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415, 416. Construction which serves to prove the general equation

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417. Any limits a and b may be taken for the integral with respect to α. These limits are those of the values of x which correspond to existing values of the function $f(x)$. Every other value of x gives a nul result for $f(x)$ 429

418. The same remark applies to the general equation

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ERRATA.

Page 9, line 28, *for* III. *read* IV.
Pages 54, 55, for k read K.
Page 189, line 2, The equation should be denoted (A).
Page 205, last line but one, for x read X.
Page 298, line 18, for $\frac{du}{dr}$ read $\frac{du}{dx}$.
Page 299, line 16, for *of* read *in*.
,, ,, last line, *read*

$$\int_0^\pi du\,\phi\,(t\sin u) = \pi\phi + tS_1\phi' + \frac{t^2}{\lfloor 2} S_2\phi'' + \&c.$$

Page 300, line 3, *for* A_2, A_4, A_6, *read* πA_2, πA_4, πA_6.
Page 407, line 12, for $d\phi$ read dp.
Page 432, line 13, read $(x-a)$.