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The Analytical Theory of Heat

First published in 1878, *The Analytical Theory of Heat* is Alexander Freeman's English translation of French mathematician Joseph Fourier's *Theorie Analytique de la Chaleur*, originally published in French in 1822. In this groundbreaking study, arguing that previous theories of mechanics advanced by such scientific greats as Archimedes, Galileo, Newton and their successors did not explain the laws of heat, Fourier set out to study the mathematical laws governing heat diffusion and proposed that an infinite mathematical series may be used to analyse the conduction of heat in solids. Known in scientific circles as the 'Fourier Series', this work paved the way for modern mathematical physics. This translation, now reissued, contains footnotes that cross-reference other writings by Fourier and his contemporaries, along with 20 figures and an extensive bibliography. This book will be especially useful for mathematicians who are interested in trigonometric series and their applications.

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THE

ANALYTICAL THEORY OF HEAT

BY

JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

BY

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PREFACE.

IN preparing this version in English of Fourier's celebrated treatise on Heat, the translator has followed faithfully the French original. He has, however, appended brief foot-notes, in which will be found references to other writings of Fourier and modern authors on the subject: these are distinguished by the initials A. F. The notes marked R. L. E. are taken from pencil memoranda on the margin of a copy of the work that formerly belonged to the late Robert Leslie Ellis, Fellow of Trinity College, and is now in the possession of St John's College. It was the translator's hope to have been able to prefix to this treatise a Memoir of Fourier's life with some account of his writings; unforeseen circumstances have however prevented its completion in time to appear with the present work.

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$C = a + 2^5b + 3^5c + 4^5d + \&c.,$	
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$\&c., \quad \quad \quad \&c.$	
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$\frac{\pi}{2} \phi(x) = \sin x \int_0^\pi da \phi(a) \sin a + \sin 2x \int_0^\pi da \phi(a) \sin 2a + \sin 3x \int_0^\pi da \phi(a) \sin 3a + \&c.,$	
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Each of the coefficients is a definite integral. We have in general

$$2\pi A = \int_{-\pi}^{+\pi} dx F(x), \quad \pi a_i = \int_{-\pi}^{+\pi} dx F(x) \cos ix,$$

and
$$\pi b_i = \int_{-\pi}^{+\pi} dx F(x) \sin ix.$$

We thus form the general theorem, which is one of the chief elements of our analysis:

$$2\pi F(x) = \sum_{i=-\infty}^{i=+\infty} \left(\cos ix \int_{-\pi}^{+\pi} da F(a) \cos ia + \sin ix \int_{-\pi}^{+\pi} da F(a) \sin ia \right),$$

$$\text{or } 2\pi F(x) = \sum_{i=-\infty}^{i=+\infty} \int_{-\pi}^{+\pi} da F(a) \cos (ix - ia). \quad . \quad . \quad . \quad . \quad . \quad . \quad 199$$

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GENERAL SOLUTION.

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$$a + bz + c \frac{z^2}{2} + d \frac{z^3}{2.3} + \&c.,$$

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$$a + \frac{ct^2}{2^2} + \frac{et^4}{2^2.4^2} + \frac{gt^6}{2^2.4^2.6^2} + \&c.,$$

is	$\frac{1}{\pi} \int_0^\pi du \phi (t \sin u).$
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OF THE FREE MOVEMENT OF HEAT IN AN INFINITE SOLID.

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$$\frac{dv}{dt} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}$$

solves the proposed problem. It cannot have a more extended integral; it is derived also from the particular value

$$v = e^{-n^2 t} \cos nx,$$

or from this:

$$v = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}},$$

which both satisfy the equation $\frac{dv}{dt} = \frac{d^2v}{dx^2}$. The generality of the integrals obtained is founded upon the following proposition, which may be regarded as self-evident. Two functions of the variables x, y, z, t are necessarily identical, if they satisfy the differential equation

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2},$$

and if at the same time they have the same value for a certain value
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$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} da f(a) \int_{-\infty}^{+\infty} dp \cos(px - pa) \dots (B) \quad . \quad . \quad ib.$$

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ERRATA.

- Page 9, line 28, *for* III. *read* IV.
Pages 54, 55, *for* k *read* K .
Page 189, line 2, The equation should be denoted (A).
Page 205, last line but one, *for* x *read* X .
Page 298, line 18, *for* $\frac{du}{dr}$ *read* $\frac{du}{dx}$.
Page 299, line 16, *for* *of* *read* *in*.
,, ,, last line, *read*
- $$\int_0^\pi du \phi(t \sin u) = \pi\phi + tS_1\phi' + \frac{t^2}{2}S_2\phi'' + \&c.$$
- Page 300, line 3, *for* A_2, A_4, A_6 , *read* $\pi A_2, \pi A_4, \pi A_6$.
Page 407, line 12, *for* $d\phi$ *read* dp .
Page 432, line 13, *read* $(x - \alpha)$.