CHAPTER I

EXPERIMENTS IN DYNAMICS

EXPERIMENT 1. **An example of conservation of angular momentum.***

1. Angular momentum of a particle about an axis. If at any instant a particle of mass m be moving with velocity v, it has momentum mv. The momentum is a vector lying along the tangent to the path of the particle at the point where the particle is at the instant considered.

Just as a force has a moment about any straight line or axis A unless its line of action either intersect A or be parallel to it, so momentum P along a given straight line has a moment about an axis A unless the straight line either intersect A or be parallel to it. To find the value of the moment of P about A we take a plane cutting A at right angles at O and project P upon this plane. If the perpendicular from O upon the projection of P be ON, and if the angle between P and the axis be θ, the moment of the projection about OA is $ON \,.\, P \sin\theta$. The momentum P has the component $P\cos\theta$ parallel to A, but this has no moment about A. Hence $ON \,.\, P\sin\theta$ is the moment of the momentum P about A.

The moment about A of the momentum mv of the particle m is often called the angular momentum of m about A.

2. Angular momentum of a system about an axis. For the purpose of the experiment we may confine ourselves to the simple case in which every particle of the system moves parallel to a fixed plane, which we take as the plane of Oxy (Fig. 1). The axis of z is perpendicular to Oxy and its positive direction is towards the reader. The three axes then form a right-handed system. In the experiment the plane Oxy is horizontal. Let P be the projection on Oxy of a particle of mass m. Let the coordinates of P relative to the fixed

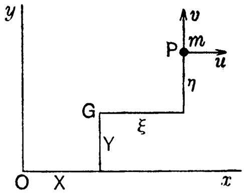

Fig. 1

* "An experiment illustrating the conservation of angular momentum," *Proc. Camb. Phil. Soc.* Vol. XXI, p. 75 (1922).

axes Ox, Oy be x, y, and let the components of the velocity of P be u, v. Then the components of the momentum of the particle are mu, mv. Hence, when counter-clockwise rotation about Oz is counted as positive, the sum of the moments about Oz of the momenta mu, mv, or the angular momentum of the particle about Oz, is $m(vx-uy)$. If H be the angular momentum about Oz of the whole system of particles,

$$H=\Sigma m(vx-uy). \qquad \text{..................(1)}$$

Let G be the projection on Oxy of the centre of gravity of the system. Let its coordinates relative to the fixed axes Ox, Oy be X, Y, and let the components of its velocity be U, V. Let $x=X+\xi$, $y=Y+\eta$, so that ξ, η are the coordinates of P relative to axes through G parallel to the fixed axes Ox, Oy. Then, since G is the centre of gravity,

$$\Sigma m\xi=0, \qquad \Sigma m\eta=0. \qquad \text{..................(2)}$$

Let $u=U+\alpha$, $v=V+\beta$, so that α, β are the components of the velocity of P relative to G. Then, since $u=dx/dt$, $v=dy/dt$, $U=dX/dt$, $V=dY/dt$, we have $\alpha=d\xi/dt$, $\beta=d\eta/dt$. But, by (2),

$$\Sigma m\,d\xi/dt=0, \qquad \Sigma m\,d\eta/dt=0,$$

and hence $$\Sigma m\alpha=0, \qquad \Sigma m\beta=0. \qquad \text{..................(3)}$$

Now, by (1),

$$H=\Sigma m\{(V+\beta)(X+\xi)-(U+\alpha)(Y+\eta)\}$$
$$=\Sigma m\{VX-UY+V\xi-U\eta+\beta X-\alpha Y+\beta\xi-\alpha\eta\}.$$

Since U, V and X, Y do not change from particle to particle, we may bring them outside the sign of summation. Thus

$$H=(VX-UY)\Sigma m+V\Sigma m\xi-U\Sigma m\eta$$
$$+X\Sigma m\beta-Y\Sigma m\alpha+\Sigma m(\beta\xi-\alpha\eta).$$

Denoting Σm, the mass of the whole system, by M and using (2) and (3), we have

$$H=M(VX-UY)+\Sigma m(\beta\xi-\alpha\eta). \qquad \text{............(4)}$$

The first term in (4) is the angular momentum about Oz of a particle of mass M placed at G and moving with G. The second term is the angular momentum of the system, for the motion relative to G, about an axis through G parallel to Oz.

3. Method. A board D (Fig. 2) is suspended by a thread which we suppose to exert no torsional control. The thread is attached to a torsion head, which is carried by a fixed support. The plane of the board is horizontal and the axis of suspension cuts the board in O. The vertical through O is the axis Oz of § 2. The inertia bar AB turns about a vertical shaft fixed to the board, the axis of the shaft passing through G, the centre of gravity of AB. A second bar C, suitably fixed to the board, acts as a counterpoise to AB. By adjusting C, the plane of D is made horizontal. By a light spring E attached to the board and operating by a string wound round a drum carried by AB, this bar can be set in motion about G relative to the board, when a thread attached to the board and holding AB in its initial position is burned. Before the thread is burned, the system is at rest. At any later time let the axis OG of the board make an angle θ with OF, its initial direction, and let the axis GA of the bar make an angle $\phi+\epsilon$ with GO, where ϵ is the angle between GA and GO before the thread is burned. Let ϕ be measured in the opposite direction to θ.

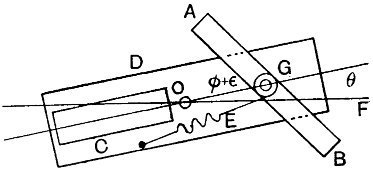

Fig. 2

With the exception of gravity and the tension of the suspension, no external forces act on the system at any time. Before the thread is burned, the system has no angular momentum about Oz. The burning of the thread does not call any external force into action and hence the angular momentum of the system about the vertical axis Oz remains zero. If the moment of inertia about Oz of the board, the counterpoise and all the other fittings *except* the bar AB be K_1, the angular momentum about Oz of this part of the system is $K_1 d\theta/dt$.

Let M be the mass of AB and let $OG=a$. Since the linear velocity of G is $a\,d\theta/dt$, the momentum of M at G is $Ma\,d\theta/dt$, and the moment of this momentum about Oz is $Ma^2 d\theta/dt$. This angular momentum is in the same direction as that of the board.

The angular velocity of AB, in the same direction as that of the board, is $d\theta/dt - d\phi/dt$, and hence, if the moment of inertia of AB about a vertical axis through G be K_2, its angular momentum about that axis is $K_2(d\theta/dt-d\phi/dt)$. Hence, by (4), the angular momentum of AB about Oz is

$$Ma^2 d\theta/dt+K_2(d\theta/dt-d\phi/dt).$$

The angular momentum of the whole system is zero, and hence

$$K_1 d\theta/dt+Ma^2 d\theta/dt+K_2(d\theta/dt-d\phi/dt)=0. \quad ......(5)$$

Since
$$\frac{d\theta}{dt}\bigg/\frac{d\phi}{dt}=\frac{d\theta}{d\phi},$$

we find, by (5), on dividing throughout by $d\phi/dt$,

$$\frac{d\theta}{d\phi}=\frac{K_2}{K_1+K_2+Ma^2}=\frac{K_2}{K_3}. \quad ....(6)$$

The quantity $K_1+K_2+Ma^2$, or K_3, is the moment of inertia about Oz of the whole system, when the bar AB is fixed relative to the board.

Since (6) holds at every instant, we have

$$\theta/\phi=K_2/K_3, \quad ....(7)$$

where θ and ϕ are, respectively, the angles turned through by the board relative to the initial line OF and by the bar relative to the board in any time, each measured from the corresponding zero.

By (5), $d\theta/dt=0$ when $d\phi/dt=0$. Hence if, at any instant, the motion of the bar relative to the board be arrested by interaction between them, the motion of the board will cease at the same instant.

Let the bar AB be removed from the board and be attached to a vertical torsion wire and be made to execute torsional vibrations, the axis of suspension agreeing with that about which AB turns when on the board. Let T_2 be the periodic time. Let T_3 be the periodic time when the *complete* system is suspended from the same wire; in this measurement AB is fixed relative to the board. Then $K_2/K_3=T_2^2/T_3^2$ and thus, by (7),

$$\theta/\phi=T_2^2/T_3^2. \quad ....(8)$$

In the experiment θ/ϕ is compared with T_2^2/T_3^2.

4. Experimental details. Fig. 3 shows some details of the apparatus. The counterpoise does not appear, as it is fixed to the part of the board which is omitted. The counterpoise and other fittings should be designed so that the axis of suspension is as nearly as possible a "principal axis." If this condition be not secured, the motion of the board will be unsteady. The ends of the torsion wire W, used in comparing K_2 with K_3, are soldered into two short cylinders 0·5 cm. in diameter. At the centre of the board is a socket H, provided with a set-screw. Into this socket fits either the rod N, by which the system is attached to the thread, or the cylinder at the end of the torsion wire. The hole in the bar AB at G is of the same diameter as that in H. By a set-screw, the bar can be secured to the torsion wire.

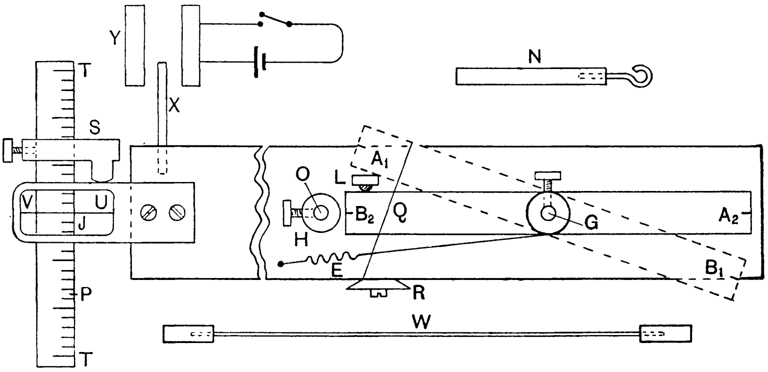

Fig. 3

The bar is held in its initial position A_1B_1 against the pull of the spring E* by a thread Q secured under the button R; this position is defined by the stop L. When the thread is burned, the bar turns round until it hits the other side of L and is arrested in the position A_2B_2. To prevent rebound, a small pellet of plasticene is placed on L as shown. Unless the plasticene be re-moulded into pellet form after each impact, the bar will rebound. Though the rebound does not vitiate the experiment, it makes the observations more difficult. The angle ϕ turned through by AB relative to the board is $\pi - A_1GB_2$. If $A_1G = r$, we have $\sin \frac{1}{2} A_1GB_2 = \frac{1}{2} A_1B_2/r$. Then

$$\phi = \pi - A_1GB_2 = \pi - 2\sin^{-1}(\tfrac{1}{2}A_1B_2/r) \text{ radians.} \quad \ldots(9)$$

* In order that the spring may have the length needed to allow the extension required, the anchorage of E is much farther from G than is suggested in Fig. 3. The thread Q may pass through an eye in the end of a short horizontal arm so fixed to the board that the part of Q between the eye and the button R is held away from the side of the board. The flame is then easily applied to the thread.

For the determination of A_1B_2, the board is placed on the table and a vertical pin is held in a clip so that its tip is just above the point A_1 on the end of the bar. The bar is then turned round into the second position shown in Fig. 3 and the distance between the tip of the pin and the point B_2 is measured.

The angle θ turned through by the board is measured by the fine wire UV, which is stretched in a metal frame carried by the board and intersects the axis Oz at right angles; it is convenient if the edges of the board be parallel to OUV. The wire moves over a horizontal scale whose edge is TT. A zero J on TT is chosen, and TT is set perpendicular to OJ. If UV cut the edge in J initially and in P finally,

$$\tan\theta = PJ/JO. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

After the observations for the angle θ have been taken, the board with its rod N is unhooked from the loop at the end of the suspending thread. A plummet is then hung from the suspending thread by a hook and is immersed in water contained in a vessel; the water damps any oscillations of the plummet. The horizontal distance between J and the axis of the plummet's thread is now easily measured.

For success, the system must be *at rest* when the thread Q is burned. The silk thread supporting the system is attached to a torsion head, which is adjusted so that UV cuts TT in some point very near J when the system is at rest. A stop S is fixed to the scale so that, when the frame touches S, UV cuts TT in J. A current sent through the coil Y attracts the small magnet X attached to the board with a force which should be only just great enough to keep the frame against the stop. The centre O of the board is then reduced to rest. A flame or a small gas jet is prepared and, when the system is at rest, the thread Q is burned. The current is stopped just before the flame is applied. The board moves round, and the reading of UV in its new position of rest at P is taken. The wire may subsequently drift very slowly from P on account of slight torsion of the silk thread or on account of draughts, and thus the reading should be taken without delay.

The supporting thread should be of *plaited* silk; *fine* fishing line may be used. If a *twisted* thread be used, it will be difficult to obtain anything like a steady zero position, since a twisted thread exerts a couple approximately proportional to the load, unless the thread be practically "unwound."

5. Practical example.

In the present (1934) apparatus the board is 56 cm. long, 8 cm. wide and 1·5 cm. thick. The inertia bar is 26 cm. long, 1·6 cm. wide and 1·6 cm. thick. The mass of the whole system is 1770 grm.

Mr J. A. Pattern, using an earlier form of apparatus, obtained the following results. Values found for JP were 7·50, 7·80, 7·50, 7·70. Mean 7·625 cm. Distance JO was 38·5 cm. Hence $\tan\theta = 7{\cdot}625/38{\cdot}5 = 0{\cdot}19805$, and

$$\theta = 11^\circ\ 12'\ 9'' = 0{\cdot}19552 \text{ radian.}$$

Distances were, $A_1G = 15{\cdot}0$ cm., $A_1B_2 = 2{\cdot}42$ cm. Hence

$$\phi = \pi - 2\sin^{-1}(1{\cdot}21/15) = 180^\circ - 9^\circ\ 15'\ 14'' = \pi - 0{\cdot}1615 \text{ radian.}$$

Thus $$\phi = 2{\cdot}9801 \text{ radians.}$$

Inertia bar was suspended by torsion wire and transits were observed.

Transit	Time mins.	Time secs.	Transit	Time mins.	Time secs.	$50T_2$ mins.	$50T_2$ secs.
0		59	50	3	43	2	44
10	1	32	60	4	16	2	44
20	2	5	70	4	48	2	43
30	2	37	80	5	21	2	44
40	3	10	90	5	53	2	43

Mean value of $50T_2$ is 163·6 secs. Hence $T_2 = 3{\cdot}272$ secs.

When whole system, including inertia bar, was suspended by the same wire, similar observations gave $50T_3 = 638{\cdot}9$ secs. Hence $T_3 = 12{\cdot}778$ secs.

For the times, we have $T_2{}^2/T_3{}^2 = 3{\cdot}272^2/12{\cdot}778^2 = 0{\cdot}06557$.

For the angles, we have $\theta/\phi = 0{\cdot}19552/2{\cdot}9801 = 0{\cdot}06561$.

Hence equation (8) is closely verified.

Experiment 2. Kater's pendulum.

6. Introduction. Since "g" is an acceleration, its determination involves the measurement of a time and a length. The "simple" pendulum of theory, with its formula $t = 2\pi\sqrt{(l/g)}$, involving a single time and a single length, would be an ideal means of finding "g." But such a pendulum cannot be made. If we use a sphere of finite size as bob, we must use a wire of sufficient strength to carry the bob and must fix some sort of knife edge to the wire. A little consideration will show that the method is beset with such difficulties that it cannot be used in an accurate determination of "g."

We therefore abandon the "simple" pendulum and use a rigid pendulum. It is impossible to determine the moment of inertia of the pendulum about any given axis by a calculation depending upon a knowledge of the geometrical form of the pendulum, for we cannot tell how the density of the material varies from point to point. We can, however, use the theorem of parallel axes* in such a way as to eliminate any uncertainty.

If we use *two* parallel knife edges *fixed* to the pendulum in such positions that the centre of gravity, G, lies *between* the knife edges in the plane containing them, we can, in theory, arrange matters so that (1) the periodic time about either knife edge is the same and (2) the distances of G from the two knife edges are different. In each case, of course, the knife edge rests on a horizontal plane. The length of the corresponding "simple equivalent pendulum" is then equal to the shortest distance between the knife edges. We cannot attain this ideal, but, if the times be very nearly equal, the necessary correction is found with ample accuracy.

7. Periodic time of a rigid pendulum. Let the mass of the pendulum be M and let G be its centre of gravity. Through G we take an axis in a direction fixed relative to the pendulum; this we call the G-axis. Let the moment of inertia of the pendulum about the G-axis be Mk^2. Let the pendulum have a knife edge parallel to the G-axis and at distance h from it, and let the pendulum oscillate through a small arc while the knife edge rests upon a horizontal plane. By the theorem of parallel axes, the moment of inertia about the knife edge is $M(k^2+h^2)$. If t be the periodic time,

$$t^2 = 4\pi^2(k^2+h^2)/gh, \quad \text{......(1)}$$

or

$$h^2 - ght^2/4\pi^2 + k^2 = 0. \quad \text{......(2)}$$

If l be the length of the simple equivalent pendulum, $l = gt^2/4\pi^2$, and thus

$$h^2 - lh + k^2 = 0. \quad \text{......(3)}$$

We are not restricted to a single position for the knife edge, for if the edge lie along *any* generating line of a cylinder of radius h described about the G-axis, t^2 will have the value given by (1).

* *Experimental Elasticity*, Note IV.

Solving (2) or (3) for h, we have

$$h=\frac{gt^2}{8\pi^2}\pm\sqrt{\left\{\left(\frac{gt^2}{8\pi^2}\right)^2-k^2\right\}}=\frac{l}{2}\pm\sqrt{\left(\frac{l^2}{4}-k^2\right)}. \quad \text{......(4)}$$

For any value of t greater than $\sqrt{(8\pi^2k/g)}$, or for any value of l greater than $2k$, there are two values of h. Since h is real, the least value of l is $2k$, and, when l has this minimum value, the distance of the knife edge from G is k, the radius of gyration of the pendulum about the G-axis.

8. Pendulum with two parallel knife edges. Let the knife edges be E, F; each is parallel to the G-axis. Let the distances of E, F from G be h_1, h_2, let t_1, t_2 be the periodic times for those knife edges and l_1, l_2 the lengths of the simple equivalent pendulums. Then

$$h_1t_1^2=4\pi^2(k^2+h_1^2)/g, \qquad h_2t_2^2=4\pi^2(k^2+h_2^2)/g.$$

By subtraction we eliminate k and obtain

$$h_1t_1^2-h_2t_2^2=4\pi^2(h_1^2-h_2^2)/g.$$

Hence
$$\frac{4\pi^2}{g}=\frac{h_1t_1^2-h_2t_2^2}{h_1^2-h_2^2}=A+B, \quad \text{..............(5)}$$

where
$$A=\frac{t_1^2+t_2^2}{2(h_1+h_2)}, \qquad B=\frac{t_1^2-t_2^2}{2(h_1-h_2)}.$$

By the use of *two* knife edges instead of one we gain the great advantage that k, whose value we cannot calculate, does not appear in (5). To find g we now have to measure the two times t_1, t_2 and the two distances h_1, h_2 of E, F from G. The times present no difficulty, but we cannot identify the position of G with any accuracy. If, however, E, F be so placed that (1) the plane through them passes also through G and (2) G lies between them, h_1+h_2 is simply the distance between the knife edges, and this can be measured accurately.

If we can make $t_2=t_1$ *without making* $h_2=h_1$, the term B will vanish and then we have

$$4\pi^2/g=t^2/(h_1+h_2),$$

where t is the common value of t_1 and t_2. We have thus recovered the simplicity of the theoretical "simple" pendulum with a single time and a single distance. In practice we cannot quite attain this ideal.

To determine the difference $h_1 - h_2$ which occurs in B we must measure h_1 and h_2 separately, since E and F are on opposite sides of G, and these measurements cannot be made with accuracy. The numerator in B is small if t_2 nearly equals t_1. If h_1 and h_2 differ widely, B will be so small compared with A, that the error due to the uncertainty in $h_1 - h_2$ will be negligible. We have therefore to secure that t_1 and t_2 are nearly equal while h_1 and h_2 are widely different. How to do this we can discover by aid of equation (4).

For a given value of l_1 we have a choice of two values of h_1 and similarly for l_2 and h_2. For h_1 we take the positive sign before the square root and for h_2 the negative sign. Then

$$h_1 - h_2 = \tfrac{1}{2}(l_1 - l_2) + \sqrt{(\tfrac{1}{4}l_1^2 - k^2)} + \sqrt{(\tfrac{1}{4}l_2^2 - k^2)},$$

and so $h_1 - h_2$ will be large even though l_1 and l_2 be nearly equal provided that both l_1 and l_2 be large compared with the minimum $2k$ or that the periodic times be much greater than the minimum $2\pi\sqrt{(2k/g)}$.

We shall fail in our purpose if, in (4), we take the same sign before the square root for both h_1 and h_2. For then we should have

$$h_1 - h_2 = \tfrac{1}{2}(l_1 - l_2) \pm \{\sqrt{(\tfrac{1}{4}l_1^2 - k^2)} - \sqrt{(\tfrac{1}{4}l_2^2 - k^2)}\},$$

and $|h_1 - h_2|$ will be small when l_1 and l_2 are nearly equal.

We can, of course, make $t_1 = t_2$ by making $h_1 = h_2 = h$, where h has any arbitrary value, but we must avoid the error of supposing that this will necessarily make $l = 2h$. We shall not have $l = 2h$ unless h_1 and h_2 be not only roots of (3) but also be equal. In this case, as shown in § 7, $l = 2k$ and $h_1 = h_2 = k$. These conditions could be secured by adjusting two massless knife edges until the periodic time about each reached a minimum value, which would be the same for each, but this process would be difficult and without any advantage.

An approximate knowledge of masses and dimensions will enable us to design a pendulum with two knife edges in such positions that h_1, h_2 are widely different although the periodic times are approximately equal. The knife edges being fixed,