ORDINARY DIFFERENTIAL EQUATIONS

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ORDINARY DIFFERENTIAL EQUATIONS

A Practical Guide

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Preface

Purpose and scope

This book explains key concepts and methods in the field of ordinary differential equations. It assumes only minimal mathematical prerequisites but, at the same time, introduces the reader to the way ordinary differential equations are used in current mathematical research and in scientific modelling. It is designed as a practical guide for students and aspiring researchers in any mathematical science - in which I include, besides mathematics itself, physics, engineering, computer science, probability theory, statistics and the quantitative side of chemistry, biology, economics and finance.

The subject of differential equations is vast and this book only deals with initial value problems for ordinary differential equations. Such problems are fundamental in modern science since they arise when one tries to predict the future from knowledge about the present. Applications of differential equations in the physical and biological sciences occupy a prominent place both in the main text and in the exercises. Numerical methods for solving differential equations are not studied in any detail, but the use of mathematical software for solving differential equations and plotting functions is encouraged and sometimes required.

How to use this book

The book should be useful for students at a range of levels and with a variety of scientific backgrounds, provided they have studied differential and integral calculus (including partial derivatives), elements of real analysis (such as $\epsilon\delta$-definitions of continuity and differentiability), complex numbers and linear algebra. It could serve as a textbook for a first course on ordinary differential equations for undergraduates on a mathematics, science, engineering or economics degree who have studied the prerequisites listed above. For such readers it offers a fairly short and steep path through the material which includes vistas onto some advanced results. The book could also be used by more advanced undergraduates or early postgraduates who would like to revisit ordinary differential equations or who, studying them for the first time, want a compact treatment that takes into account their mathematical maturity.

Regardless of their background, readers should be aware that the chapters do not necessarily have to be studied in the order in which they are numbered. Having studied first order equations in Chapter 1, the reader can go on to study the theory underpinning general systems of differential equations in Chapter 2. Alternatively, if the mathematical generality of Chapter 2 seems too daunting, the reader could first study Chapter 3, which deals with specific second order equations and their applications in physics. This would mean taking certain recipes on trust, but would allow the reader to build up more experience with differential equations before mastering the general theory contained in Chapter 2.

The exercises form an integral part of the text, and the reader is strongly encouraged to attempt them *all*. They range in difficulty from routine to challenging, and some introduce or develop key ideas which are only mentioned in the main text. Solutions are not included, and it is quite likely that the reader will initially get stuck on some of the problems. This is entirely intended. Being stuck is an essential part of doing mathematics, and every mathematical training should include opportunities for getting stuck - and thus for learning how to get unstuck. One simple strategy is to move on and to return to the problem with a fresh mind later. Another is to discuss the problem with a fellow student - face to face or in ‘cyberspace’.

The last chapter of the book consists of five projects. The first four are authentic in the sense that they address problems which arose in real and recent research. Readers, particularly those who have studied differential equations before, are encouraged to attempt (parts of) these projects before reaching the end of the book. In that way, the projects provide motivation for some of the theory covered in the book - and assure the reader that the techniques introduced are not mere finger exercises. If possible, the projects could be tackled by a small group of students working as a team. The last project is a guided proof of the existence and uniqueness theorem for initial value problems. The material there could also serve as the basis for a seminar on this proof.

Acknowledgements

This book grew out of courses on ordinary differential equations which I taught at Heriot-Watt University in Edinburgh and at the African Institute for Mathematical Sciences (AIMS) in Muizenberg, South Africa. In writing up my lecture notes I have tried to capture the spirit of the course at AIMS, with its strong emphasis on problem solving, team work and interactive teaching.

I am indebted to a number of people who contributed to this book. Ken Brown's notes on differential equations provided the foundations for my course on the subject at Heriot-Watt. Some of the pictures and type-set notes produced by Robert Weston for the same course appear, in recycled form, in Chapter 4. Aziz Ouhinou at AIMS led the seminar on the Picard-Lindelöf Theorem which evolved into Project 5. The students at AIMS whom I had the privilege of teaching during the past six years have profoundly shaped this book through their enthusiasm for learning, their mathematical intelligence and their probing questions. I am deeply grateful to them.

Notation and units

The mathematical notation in this book is mostly standard. The abbreviation 'iff' is occasionally used for 'if and only if'. The letter I denotes the identity matrix in Chapters 2 and 4. SI units are used for applications in physics, so distance is measured in metres (m), time in seconds (s), mass in kilograms (kg) and force in Newtons (N).