1
First order differential equations

1.1 General remarks about differential equations

1.1.1 Terminology

A differential equation is an equation involving a function and its derivatives. An example which we will study in detail in this book is the pendulum equation

$$\frac{d^2x}{dt^2} = -\sin(x), \tag{1.1}$$

which is a differential equation for a real-valued function x of one real variable t. The equation expresses the equality of two *functions*. To make this clear we could write (1.1) more explicitly as

$$\frac{d^2x}{dt^2}(t) = -\sin(x(t)) \qquad \text{for all } t \in \mathbb{R},$$

but this would lead to very messy expressions in more complicated equations. We therefore often suppress the independent variable.

When formulating a mathematical problem involving an equation, we need to specify *where* we are supposed to look for solutions. For example, when looking for a solution of the equation $x^2 + 1 = 0$ we might require that the unknown x is a real number, in which case there is no solution, or we might allow x to be complex, in which case we have the two solutions $x = \pm i$. In trying to solve the differential equation (1.1) we are looking for a twice-differentiable function $x : \mathbb{R} \to \mathbb{R}$. The set of all such functions is very big (bigger than the set of all real numbers, in a sense that can be made precise by using the notion of cardinality), and this is one basic reason why, generally speaking, finding solutions of differential equations is not easy.

Differential equations come in various forms, which can be classified as follows. If only derivatives of the unknown function with respect to one variable appear, we call the equation an *ordinary differential equation*, or ODE for short. If the function depends on several variables, and if partial derivatives with respect to at least two variables appear in the equation, we call it a *partial differential equation*, or PDE for short. In both cases, the order of the differential equation is the order of the highest derivative occurring in it. The independent variable(s) may be real or complex, and the unknown function may take values in the real or complex numbers, or in $\mathbb{R}^n$ of $\mathbb{C}^n$ for some positive integer n. In the latter case we can also think of the differential equation as a set of differential equations for each of the n components. Such a set is called a *system of differential equations of dimension* n. Finally we note that there may be parameters in a differential equation, which play a different role from the variables. The difference between parameters and variables is usually clear from the context, but you can tell that something is a parameter if no derivatives with respect to it appear in the equation. Nonetheless, solutions still depend on these parameters. Examples illustrating the terms we have just introduced are shown in Fig. 1.1.

In this book we are concerned with ordinary differential equations for functions of one real variable and, possibly, one or more parameters. We mostly consider real-valued functions but complex-valued functions also play an important role.

1.1.2 Approaches to problems involving differential equations

Many mathematical models in the natural and social sciences involve differential equations. Differential equations also play an important role in many branches of pure mathematics. The following system of ODEs for real-valued functions a, b and c of one

$$\frac{dy}{dx} - xy = 0$$

first order ordinary differential equation

$$\frac{du}{dt} = -v, \quad \frac{dv}{dt} = u$$

first order system of ordinary differential equations, dimension two

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3$$

second order ordinary differential equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

second order partial differential equation with parameter c

Fig. 1.1. Terminology for differential equations

real variable r arises in differential geometry:

$$\begin{aligned}
\frac{da}{dr} &= \frac{1}{2rc}(a^2 - (b-c)^2) \\
\frac{db}{dr} &= \frac{b}{2acr}(b^2 - (a-c)^2) \\
\frac{dc}{dr} &= \frac{1}{2ra}(c^2 - (a-b)^2),
\end{aligned}$$

One is looking for solutions on the interval $[\pi, \infty)$ subject to the initial conditions

$$a(\pi) = 0, \quad b(\pi) = \pi, \quad c(\pi) = -\pi.$$

At the end of this book you will be invited to study this problem as a part of an extended project. At this point, imagine that a colleague or friend had asked for your help in solving the above equations. What would you tell him or her? What are the questions that need to be addressed, and what methods do you know for coming up with answers? Try to write down some ideas before looking at the following list of issues and approaches.

(a) **Is the problem well-posed?** In mathematics, a problem is called well-posed if it has a solution, if that solution is unique and if the solution depends continuously on the data given in the problem, in a suitable sense. When a differential equation has one solution, it typically has infinitely many. In order to obtain a well-posed problem we therefore need to complement the differential equation with additional requirements on the solution. These could be initial conditions (imposed on the solution and its derivatives at one point) or boundary conditions (imposed at several points).

(b) **Are solutions stable?** We would often like to know how a given solution changes if we change the initial data by a small amount.

(c) **Are there explicit solutions?** Finding explicit solutions in terms of standard functions is only possible in rare lucky circumstances. However, when an explicit formula for a general solution can be found, it usually provides the most effective way of answering questions related to the differential equation. It is therefore useful to know the types of differential equation which can be solved exactly.

(d) **Can we find approximate solutions?** You may be able to solve a simpler version of the model exactly, or you may be able to give an approximate solution of the differential equation. In all approximation methods it is important to have some control over the accuracy of the approximation.

(e) **Can we use geometry to gain qualitative insights?** It is often possible to derive general, qualitative features of solutions without solving the differential equation. These could include asymptotic behaviour (what happens to the solution for large r?) and stability discussed under (b).

(f) **Can we obtain numerical solutions?** Many numerical routines for solving differential equations can be downloaded from open-source libraries like SciPy. Before using them, check if the problem you are trying to solve is well-posed. Having some insight into approximate or qualitative features of the solution usually helps with the numerical work.

In this text we will address all of these issues. We begin by looking at first order differential equations.

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1.2 Exactly solvable first order ODEs

1.2.1 Terminology

The most general first order ODE for a real-valued function x of one real variable t is of the form

$$F\left(t, x, \frac{dx}{dt}\right) = 0, \tag{1.2}$$

for some real-valued function F of three variables. The function x is a solution if it is defined at least on some interval $I \subset \mathbb{R}$ and if

$$F\left(t, x(t), \frac{dx}{dt}(t)\right) = 0 \text{ for all } t \in I.$$

We call a first order ODE *explicit* if it can be written in terms of a real-valued function f of two variables as

$$\frac{dx}{dt} = f(t, x). \tag{1.3}$$

Otherwise, the ODE is called *implicit*. Before we try to understand the general case, we consider some examples where solutions can be found by elementary methods.

1.2.2 Solution by integration

The simplest kind of differential equation can be written in the form

$$\frac{dx}{dt} = f(t),$$

where f is a real-valued function of one variable, which we assume to be continuous. By the fundamental theorem of calculus we can find solutions by integration

$$x(t) = \int f(t)dt.$$

The right hand side is an indefinite integral, which contains an arbitrary constant. As we shall see, solutions of first order differential equations are typically only determined up to an arbitrary constant. Solutions to first order ODEs which contain an arbitrary constant are called *general solutions*.

Exercise 1.1 Revise your integration by finding general solutions of the following ODEs. Where are these solutions valid?

$$\text{(i)}\ \frac{dx}{dt} = \sin(4t-3), \qquad \text{(ii)}\ \frac{dx}{dt} = \frac{1}{t^2-1}.$$

1.2.3 Separable equations

Let us find the general solution of the following slightly more complicated equation:

$$\frac{dx}{dt} = 2tx^2. \tag{1.4}$$

It can be solved by separating the variables, i.e., by bringing all the t-dependence to one side and all the x-dependence to the other, and integrating:

$$\int \frac{1}{x^2}dx = \int 2tdt,$$

where we need to assume that $x \neq 0$. Integrating once yields

$$-\frac{1}{x} = t^2 + c,$$

with an arbitrary real constant c. Hence, we obtain a one-parameter family of solutions of (1.4):

$$x(t) = -\frac{1}{t^2+c}. \tag{1.5}$$

In finding this family of solutions we had to assume that $x \neq 0$. However, it is easy to check that the constant function $x(t) = 0$ for all t also solves (1.4). It turns out that the example is typical of the general situation: in separating variables we may lose constant solutions, but these can be recovered easily by inspection. A precise formulation of this statement is given in Exercise 1.8, where you are asked to prove it using an existence and uniqueness theorem which we discuss in Section 1.3.

The general solution of a first order ODE is really the family of all solutions of the ODE, usually parametrised by a real number. A first order ODE by itself is therefore, in general, not a well-posed problem in the sense of Section 1.1 because it does not have a unique solution. In the elementary examples of ODEs in Exercise

1.1 and also in the ODE (1.4) we can obtain a well-posed problem if we impose an initial condition on the solution. If we demand that a solution of (1.4) satisfies $x(0) = 1$ we obtain the unique answer $x(t) = \frac{1}{1-t^2}$. If, on the other hand, we demand that a solution satisfies $x(0) = 0$ then the constant function $x(t) = 0$ for all t is the only possibility. The combination of a first order ODE with a specification of the value of the solution at one point is called an *initial value problem*.

The method of solving a first order ODE by separating variables works, at least in principle, for any equation of the form

$$\frac{dx}{dt} = f(t)g(x),$$

where f and g are continuous functions. The solution $x(t)$ is determined implicitly by

$$\int \frac{1}{g(x)} dx = \int f(t) dt. \tag{1.6}$$

In general, it may be impossible to express the integrals in terms of elementary functions and to solve explicitly for x.

1.2.4 Linear first order differential equations

If the function f in equation (1.3) is a sum of terms which are either independent of x or linear in x, we call the equation linear. Consider the following example of an initial value problem for a linear first order ODE:

$$\frac{dx}{dt} + 2tx = t \qquad x(0) = 1. \tag{1.7}$$

First order linear equations can always be solved by using an *integrating factor*. In the above example, we multiply both sides of the differential equation by $\exp(t^2)$ to obtain

$$e^{t^2}\frac{dx}{dt} + 2te^{t^2}x = te^{t^2}. \tag{1.8}$$

Now the left hand side has become a derivative, and the equation (1.8) can be written as

$$\frac{d}{dt}\left(e^{t^2}x\right) = te^{t^2}.$$

Integrating once yields $xe^{t^2} = \frac{1}{2}e^{t^2} + c$, and hence the general solution

$$x(t) = \frac{1}{2} + ce^{-t^2}.$$

Imposing $x(0) = 1$ implies $c = \frac{1}{2}$ so that the solution of (1.7) is

$$x(t) = \frac{1}{2}\left(1 + e^{-t^2}\right).$$

General linear equations of the form

$$\frac{dx}{dt} + a(t)x = b(t), \tag{1.9}$$

where a and b are continuous functions of t, can be solved using the integrating factor

$$I(t) = e^{\int a(t)dt}. \tag{1.10}$$

Since the indefinite integral $\int a(t)dt$ is only determined up to an additive constant, the integrating factor is only determined up to a multiplicative constant: if $I(t)$ is an integrating factor, so is $C \cdot I(t)$ for any non-zero real number C. In practice, we make a convenient choice. In the example above we had $a(t) = 2t$ and we picked $I(t) = \exp(t^2)$. Multiplying the general linear equation (1.9) by $I(t)$ we obtain

$$\frac{d}{dt}\left(I(t)x(t)\right) = I(t)b(t).$$

Now we integrate and solve for $x(t)$ to find the general solution. In Section 2.5 we will revisit this method as a special case of the method of variation of the parameters.

1.2.5 Exact equations

Depending on the context, the independent variable in an ODE is often called t (particularly when it physically represents time), sometimes x (for example when it is a spatial coordinate) and sometimes another letter of the Roman or Greek alphabet. It is best not to get too attached to any particular convention. In the following example, the independent real variable is called x, and the real-valued function that we are looking for is called y. The differential equation governing y as a function of x is

$$(x + \cos y)\frac{dy}{dx} + y = 0. \tag{1.11}$$

Re-arranging this as

$$(x\frac{dy}{dx} + y) + \cos y\frac{dy}{dx} = 0,$$

we note that

$$(x\frac{dy}{dx} + y) = \frac{d}{dx}(xy), \quad \text{and} \quad \cos y\frac{dy}{dx} = \frac{d}{dx}\sin y.$$

If we define $\psi(x, y) = xy + \sin y$ then (1.11) can be written

$$\frac{d}{dx}\psi(x, y(x)) = 0 \tag{1.12}$$

and is thus solved by

$$\psi(x, y(x)) = c, \tag{1.13}$$

for some constant c.

Equations which can be written in the form (1.12) for some function ψ are called *exact*. It is possible to determine whether a general equation of the form

$$a(x, y)\frac{dy}{dx}(x) + b(x, y) = 0, \tag{1.14}$$

for differentiable functions $a, b : \mathbb{R}^2 \to \mathbb{R}$, is exact as follows. Suppose equation (1.14) were exact. Then we should be able to write it in the form (1.12) for a twice-differentiable function $\psi : \mathbb{R}^2 \to \mathbb{R}$. However, by the chain rule,

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial\psi}{\partial y}\frac{dy}{dx}(x) + \frac{\partial\psi}{\partial x},$$

so that (1.14) is exact if there exists a twice-differentiable function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with

$$\frac{\partial\psi}{\partial y}(x, y) = a(x, y) \quad \text{and} \quad \frac{\partial\psi}{\partial x}(x, y) = b(x, y).$$

Since $\dfrac{\partial^2\psi}{\partial x\partial y} = \dfrac{\partial^2\psi}{\partial y\partial x}$ for twice-differentiable functions we obtain a necessary condition for the existence of the function ψ:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}. \tag{1.15}$$

The equation (1.11) is exact because $a(x,y) = x + \cos y$ and $b(x,y) = y$ satisfy (1.15):

$$\frac{\partial}{\partial x}(x + \cos y) = 1 = \frac{\partial y}{\partial y}.$$

To find the function ψ systematically, we need to solve

$$\frac{\partial \psi}{\partial y} = x + \cos y, \qquad \frac{\partial \psi}{\partial x} = y. \tag{1.16}$$

As always in solving simultaneous equations, we start with the easier of the two equations; in this case this is the second equation in (1.16), which we integrate with respect to x to find $\psi(x,y) = xy + f(y)$, where f is an unknown function. To determine it, we use the first equation in (1.16) to derive $f'(y) = \cos y$, which is solved by $f(y) = \sin y$. Thus

$$\psi(x,y) = xy + \sin y,$$

leading to the general solution given in (1.13).

It is sometimes possible to make a non-exact equation exact by multiplying with a suitable integrating factor. However, it is only possible to give a recipe for computing the integrating factor in the linear case. In general one has to rely on inspired guesswork.

1.2.6 Changing variables

Some ODEs can be simplified and solved by changing variables. We illustrate how this works by considering two important classes. **Homogeneous ODEs** are equations of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{1.17}$$

such as

$$\frac{dy}{dx} = \frac{2xy}{x^2 + y^2} = \frac{2(\frac{y}{x})}{1 + (\frac{y}{x})^2}. \tag{1.18}$$