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978-1-107-69681-5 - Theory of Differential Equations: Part III: Ordinary

Linear Equations: Vol. IV

Andrew Russell Forsyth

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