THEORY

OF

DIFFERENTIAL EQUATIONS.

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PART III.

ORDINARY LINEAR EQUATIONS.

BY

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PREFACE.

THE present volume, constituting Part III of this work, deals with the theory of ordinary linear differential equations. The whole range of that theory is too vast to be covered by a single volume; and it contains several distinct regions that have no organic relation with one another. Accordingly, I have limited the discussion to the single region specially occupied by applications of the theory of functions; in imposing this limitation, my wish has been to secure a uniform presentation of the subject.

As a natural consequence, much is omitted that would have been included, had my decision permitted the devotion of greater space to the subject. Thus the formal theory, in its various shapes, is not expounded, save as to a few topics that arise incidentally in the functional theory. The association with homogeneous forms is indicated only slightly. The discussion of combinations of the coefficients, which are invariantive under all transformations that leave the equation linear, of the associated equations that are covariantive under these transformations, and of the significance of these invariants and covariants, is completely omitted. Nor is any application of the theory of groups, save in a single functional investigation, given here. The student, who wishes to consider these subjects, and others that have been passed by, will find them in Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, in treatises such as Picard's *Cours d'Analyse*, and in many of the memoirs quoted in the present volume.

In preparing the volume, I have derived assistance from the two works just mentioned, as well as from the uncompleted work by the late Dr Thomas Craig. But, as will be seen from the references in the text, my main assistance has been drawn from the numerous memoirs contributed to learned journals by various pioneers in the gradual development of the subject.

Within the limitations that have been imposed, it will be seen that much the greater part of the volume is assigned to the theory of equations which have uniform coefficients. When coefficients are not uniform, the difficulties in the discussion are grave: the principal characteristics of the integrals of such an equation have, as yet, received only slight elucidation. On this score, it will be sufficient to mention equations having algebraic coefficients: nearly all the characteristic results that have been obtained are of the nature of existence-theorems, and little progress in the difficult task of constructing explicit results has been made.

Moreover, I have dealt mainly with the general theory and have abstained from developing detailed properties of the functions defined by important particular equations. The latter have been used as illustrations; had they been developed in fuller detail than is given, the investigations would soon have merged into discussions of the properties of special functions. Instances of such transition are provided in the functions, defined by the hypergeometric equation and by the modern form of Lamé's equation respectively.

A brief summary of the contents will indicate the actual range of the volume. In the first Chapter, the synectic integrals of a linear equation, and the conditions of their uniqueness, are investigated. The second Chapter discusses the general character of a complete system of integrals near a singularity of the equation. Chapters III, IV, and V are concerned with equations, which have their integrals of the type called regular; in particular, Chapter V contains those equations the integrals of which are algebraic functions of the variable. In Chapter VI, equations are considered which have only some of their integrals of the regular type; the influence of such integrals upon the reducibility of their equation is indicated. Chapter VII is occupied with the determination of integrals which, while not regular, are irregular of specified types called normal and subnormal; the functional significance of such integrals is established, in connection with Poincaré's development of Laplace's solution in the form of a definite integral. Chapter VIII is devoted to equations, the integrals of which do not belong to any of the preceding types; the method of converging infinite determinants is used to obtain the complete solution for any such equation. Chapter IX relates to those equations, the coefficients of which are uniform periodic functions of the variable: there are two classes, according as the periodicity is simple or double. The final Chapter deals with equations having algebraic coefficients; it contains a brief general sketch of Poincaré's association of such equations with automorphic functions.

In the revision of the proof-sheets, I have received valuable assistance from three of my friends and former pupils, Mr. E. T. Whittaker, M.A., and Mr. E. W. Barnes, M.A., Fellows of Trinity College, Cambridge, and Mr. R. W. H. T. Hudson, M.A., Fellow of St John's College, Cambridge; I gratefully acknowledge the help which they have given me.

And I cannot omit the expression of my thanks to the Staff of the University Press, for the unfailing courtesy and readiness with which they have lightened my task during the printing of the volume.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE,
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