CHAPTER I.

Linear Equations; Existence of Synectic Integrals: Fundamental Systems.

1. The course of the preceding investigations has made it manifest that the discussion of the properties of functions, which are defined by ordinary differential equations of a general type, rapidly increases in difficulty with successive increase in the order of the equations. Indeed, a stage is soon reached where the generality of form permits the deduction of no more than the simplest properties of the functions. Special forms of equations can be subjected to special treatment; but, when such special forms conserve any element of generality, complexity and difficulty arise for equations of any but the lowest orders. There is one exception to this broad statement; it is constituted by ordinary equations which are linear in form. They can be treated, if not in complete generality, yet with sufficient fulness to justify their separate discussion; and accordingly, the various important results relating to the theory of ordinary linear differential equations constitute the subject-matter of the present Part of this Treatise.

Some classes of linear equations have received substantial consideration in the construction of the customary practical methods used in finding solutions. One particular class is composed of those equations which have constants as the coefficients of the dependent variable and its derivatives. There are, further, equations associated with particular names, such as Legendre, Bessel, Lamé; there are special equations, such as those of the hypergeometric series and of the quarter-period in the Jacobian theory of elliptic functions. The formal solutions of such equations can be regarded as known; but so long as the investigation is restricted to the practical construction of the respective series adopted for the solutions, no indication of the range, over which the deduced solution is valid, is thereby given. It is the aim of the general theory, as applied to such equations, to reconstruct the various methods of proceeding to a solution, and to shew why the isolated rules, that seem so sourceless in practice, actually prove effective. In prosecuting this aim, it will be necessary to revise for linear equations all the customarily accepted results, so as to indicate their foundation, their range of validity, and their significance.

For the most part, the equations considered will be kept as general as possible within the character assigned to them. But from time to time, equations will be discussed, the functions defined by which can be expressed in terms of functions already known; such instances, however, being used chiefly as illustrations. For all equations, it will be necessary to consider the same set of problems as present themselves for consideration in the discussion of unrestricted ordinary equations of the lowest orders: the existence of an integral, its uniqueness as determined by assigned conditions, its range of existence, its singularities (as regards position and nature), its behaviour in the vicinity of any singularity, and so on: together with the converse investigation of the limitations to be imposed upon the form of the equation in order to secure that functions of specified classes or types may be solutions. As is usual in discussions of this kind, the variables and the parameters will be assumed to be complex. It is true that, for many of the simpler applications to mechanics and physics, the variables and the parameters are purely real; but this is not the case with all such applications, and instances occur in which the characteristic equations possess imaginary or complex parameters or variables. Quite independently of this latter fact, however, it is desirable to use complex variables in order to exhibit the proper relation of functional variation.

2. Let z denote the independent variable, and w the dependent variable; z and w varying each in its own plane. The differential equation is considered *linear*, when it contains no term of order higher than the first in w and its derivatives; and a linear equation is called *homogeneous*, when it contains no term independent of w

and its derivatives. By a well-known formal result*, the solution of an equation that is not homogeneous can be deduced, merely by quadratures, from the solution of the equation rendered homogeneous by the omission of the term independent of w and its derivatives; and therefore it is sufficient, for the purposes of the general investigation, to discuss homogeneous linear equations. The coefficients may be uniform functions of z, either rational or transcendental; or they may be multiform functions of z, the simplest instance being that in which they are of a form $\phi(s, z)$, where ϕ is rational in s and z, and s is an algebraic function of z. Examples of each of these classes will be considered in turn. The coefficients will have singularities and (it may be) critical points; all of these are determinable for a given equation by inspection, being fixed points which are not affected by any constants that may arise in the integration. Such points will be found to include all the singularities and the critical points of the integrals of the equation; in consequence, they are frequently called *the singularities of the equation*. Accordingly, the differential equation, assumed to be of order m, can be taken in the form

$$\frac{d^m w}{dz^m} = p_1 \frac{d^{m-1} w}{dz^{m-1}} + p_2 \frac{d^{m-2} w}{dz^{m-2}} + \dots + p_m w,$$

where the coefficients $p_1, p_2, \dots, p_m$ are functions of z. In the earlier investigations, and until explicit statement to the contrary is made, it will be assumed that these functions of z are uniform within the domain considered; that their singularities are isolated points, so that any finite part of the plane contains only a limited number of them: and that all these singularities (if any) for finite values of z are poles of the coefficients, so that their only essential singularity (if any) must be at infinity. Let ζ denote any point in the plane which is ordinary for all the coefficients p; and let a domain of ζ be constructed by taking all the points z in the plane, such that

$$|z - \zeta| \leqslant |a - \zeta|,$$

where a is the nearest to ζ among all the singularities of all the coefficients. Then within this domain (but not on its boundary) we have

$$p_s = P_s(z - \zeta), \qquad (s = 1, 2, \dots, m),$$

* See my *Treatise on Differential Equations*, § 75.

where P_s denotes a regular function of $z-\zeta$, which generally is an infinite series of powers of $z-\zeta$ converging within the domain of ζ. An integral of the equation existing in this domain is uniquely settled by the following theorem:—

*In the domain of an ordinary point ζ, the differential equation possesses an integral, which is a regular function of $z-\zeta$ and, with its first $m-1$ derivatives, acquires arbitrarily assigned values when $z=\zeta$; and this integral is the only regular function of $z-\zeta$ in the specified domain, which satisfies the equation and fulfils the assigned conditions**.

The integral thus obtained will be called† the *synectic* integral.

Synectic Integrals.

3. The existence of an integral which is a holomorphic function of $z-\zeta$ within the domain will first be established.

Let r be the radius of the domain of ζ; let $M_1, \ldots, M_m$ denote quantities not less than the maximum values of $|p_1|, \ldots, |p_m|$ respectively, for points within the domain; and let dominant functions $\phi_1, \ldots, \phi_m$, defined by the expressions

$$\phi_s = \frac{M_s}{1-\dfrac{z-\zeta}{r}}, \qquad (s=1, \ldots, m),$$

be constructed. Then‡

$$\left|\frac{d^\alpha p_s}{dz^\alpha}\right|_{z=\zeta} \leqslant \left|\frac{d^\alpha \phi_s}{dz^\alpha}\right|_{z=\zeta},$$

for every positive integer α. The dominant functions ϕ are used to construct a dominant equation

$$\frac{d^m u}{dz^m} = \phi_1 \frac{d^{m-1}u}{dz^{m-1}} + \phi_2 \frac{d^{m-2}u}{dz^{m-2}} + \ldots + \phi_m u,$$

which is considered concurrently with the given equation.

* The conditions, as to the arbitrarily assigned values to be acquired at ζ by w and its derivatives, are called the *initial conditions*; the values are called the *initial values*.

† As it is a regular function of the variable, it would have been proper to call it the regular integral. This term has however been appropriated (see Chapter III, § 29) to describe another class of integrals of linear equations; as the use in this other connection is now widespread, confusion would result if the use were changed.

‡ See my *Theory of Functions*, 2nd edn., § 22: quoted hereafter as *T. F.*

Any function which is regular in the domain of ζ can be expressed as a converging series of powers of $z-\zeta$; and the coefficients, save as to numerical factors, are the values of the various derivatives of the function at ζ. Accordingly, if there is an integral w which is a regular function of $z-\zeta$, it can be formed when the values of all the derivatives of w at ζ are known. To $w, \dfrac{dw}{dz}, \ldots, \dfrac{d^{m-1}w}{dz^{m-1}}$, the arbitrary values specified in the initial conditions are assigned. All the succeeding derivatives of w can be deduced from the differential equation in the form

$$\frac{d^\alpha w}{dz^\alpha} = A_{\alpha 1}\frac{d^{m-1}w}{dz^{m-1}} + A_{\alpha 2}\frac{d^{m-2}w}{dz^{m-2}} + \ldots + A_{\alpha m}w,$$

(for $\alpha = m$, $m+1$, ... ad inf.), by processes of differentiation, addition, and multiplication: as the coefficient of the highest derivative of w in the equation (and in every equation deduced from it by differentiation) is unity, new critical points are not introduced by these processes, so that all the coefficients A are regular within the domain of ζ.

The successive derivatives of u are similarly expressible in the form

$$\frac{d^\alpha u}{dz^\alpha} = B_{\alpha 1}\frac{d^{m-1}u}{dz^{m-1}} + B_{\alpha 2}\frac{d^{m-2}u}{dz^{m-2}} + \ldots + B_{\alpha m}u,$$

(for $\alpha = m$, $m+1$, ... ad inf.), obtained in the same way as the equation for the derivatives of w. The coefficients B have the same form as the coefficients A, and can be deduced from them by changing the quantities p and their derivatives into the quantities ϕ and their derivatives respectively.

The values of the derivatives of w and u at ζ are required. When $z=\zeta$, all the terms in each quantity B are positive; on account of the relation between the derivatives of the quantities p and ϕ, it follows that

$$B_{\alpha s} \geqslant |A_{\alpha s}|, \qquad (s = 1, \ldots, m),$$

when $z=\zeta$. Let the initial values of $|w|, \left|\dfrac{dw}{dz}\right|, \ldots, \left|\dfrac{d^{m-1}w}{dz^{m-1}}\right|$, when $z=\zeta$, be assigned as the values of $u, \dfrac{du}{dz}, \ldots, \dfrac{d^{m-1}u}{dz^{m-1}}$ when $z=\zeta$; then

$$\left|\frac{d^\alpha w}{dz^\alpha}\right| \leqslant \frac{d^\alpha u}{dz^\alpha},$$

when $z = \zeta$, for the values $m, m+1, \dots$ of α. If the series

$$(u) + (z-\zeta)\left(\frac{du}{dz}\right) + \frac{(z-\zeta)^2}{2\,!}\left(\frac{d^2u}{dz^2}\right) + \dots$$

converges, where $\left(\frac{d^\alpha u}{dz^\alpha}\right)$ denotes the value of $\frac{d^\alpha u}{dz^\alpha}$ when $z = \zeta$, the series

$$(w) + (z-\zeta)\left(\frac{dw}{dz}\right) + \frac{(z-\zeta)^2}{2\,!}\left(\frac{d^2w}{dz^2}\right) + \dots,$$

where $\left(\frac{d^\alpha w}{dz^\alpha}\right)$ denotes the value of $\frac{d^\alpha w}{dz^\alpha}$ when $z = \zeta$, also converges; it then represents a regular function of $z - \zeta$ which, after the mode of formation of its coefficients, satisfies the differential equation.

We therefore proceed to consider the convergence of the series for u, obtained as a purely formal solution of the dominant equation. To obtain explicit expressions for the various coefficients in this series, let $z - \zeta = rx$, taking x as the new independent variable. Points within the domain of ζ are given by $|x| < 1$; and the dominant equation becomes

$$(1-x)\frac{d^m u}{dx^m} = \sum_{s=1}^{m} M_s r^s \frac{d^{m-s}u}{dx^{m-s}}.$$

When the series for u, taken in the form

$$u = \sum_{\alpha=0}^{\infty} b_\alpha x^\alpha,$$

is substituted in the equation which then becomes an identity, a comparison of the coefficients of x^k on the two sides leads to the relation

$$(m+k)\,!\,b_{m+k}$$
$$= (m+k-1)\,!\,(k + M_1 r)\,b_{m+k-1} + \sum_{s=2}^{m}(m+k-s)\,!\,M_s r^s b_{m+k-s},$$

holding for all positive integer values of k.

This relation shews that all the coefficients b are expressible linearly and homogeneously in terms of $b_0, b_1, \dots, b_{m-1}$: and that, as the first m of these coefficients have been made equal to the moduli of the m arbitrary quantities in the initial conditions for w and therefore are positive, all the coefficients b are positive. Hence

$$b_{m+k} > \frac{k + M_1 r}{k+m}\,b_{m+k-1}.$$

By the initial definition of M_1, it was taken to be not less than the maximum value of $|p_1|$ within the domain of ζ; it can therefore be chosen so as to secure that $M_1 r > m$. Assuming this choice made, we then have

$$b_{m+k} > b_{m+k-1},$$

so that the successive coefficients increase.

From the difference-equation satisfied by the coefficients b, it follows that

$$\frac{b_{m+k}}{b_{m+k-1}} = \frac{k + M_1 r}{k+m} + \sum_{s=2}^{m} \frac{(m+k-s)!}{(m+k)!} M_s r^s \frac{b_{m+k-s}}{b_{m+k-1}}.$$

So far as regards the $m-1$ terms in the summation, the ratio $b_{m+k-s} \div b_{m+k-1}$ is less than unity for each of them; $M_s r^s$ is finite for each of them; and $(m+k-s)! \div (m+k)!$ is zero for each of them, in the limit when k is made infinite. Hence we have

$$\operatorname*{Lim}_{k=\infty} \frac{b_{m+k}}{b_{m+k-1}} = 1,$$

and therefore

$$\operatorname*{Lim}_{l=\infty} \frac{b_{l+1}|x|^{l+1}}{b_l |x|^l} = |x|$$
$$< 1,$$

for points within the domain of ζ, so that* the series

$$\sum_{\alpha=0}^{\infty} b_\alpha x^\alpha$$

converges within the domain of ζ. The convergence is not established for the boundary, so that it can be affirmed only for points within the domain; it holds for all arbitrary positive values assigned to $b_0, b_1, \ldots, b_{m-1}$.

It therefore follows that, at all points within the domain of ζ, a regular function of $z-\zeta$ exists which satisfies the original differential equation for w, and, with its first $m-1$ derivatives, acquires at ζ arbitrarily assigned values.

4. Now that the existence of a synectic integral is established, the explicit expression of the integral in the form of a power-series in $z-\zeta$, this series being known to converge, can be obtained

* Chrystal's *Algebra*, vol. II, p. 121.

directly from the equation. As ζ is an ordinary point for each of the coefficients p, we have

$$p_s = P_s(z-\zeta), \qquad (s = 1, 2, \ldots, m),$$

where P_s denotes a regular function of $z-\zeta$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ be the arbitrary values assigned to $w, \dfrac{dw}{dz}, \ldots, \dfrac{d^{m-1}w}{dz^{m-1}}$, when $z=\zeta$; and take

$$w = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}(z-\zeta)^n,$$

which manifestly satisfies the initial conditions. In order that this may satisfy the equation, it must make the equation an identity when the expression is substituted therein. When the substitution is effected, and the coefficients of $(z-\zeta)^s$ on the two sides of the identity are equated, we have a relation of the form

$$\frac{\alpha_{m+s}}{s!} = A_{m+s},$$

where A_{m+s} is a linear homogeneous function of the coefficients α_κ, such that $\kappa < m+s$, and is also linear in the coefficients in the quantities $P_1(z-\zeta), \ldots, P_m(z-\zeta)$; and the relation is valid for $s = 0, 1, 2, \ldots$, ad inf. Using the relation for these values of s in succession, we find $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ expressed (in each instance, after substitution of the values of the coefficients which belong to earlier values of s) as a linear homogeneous function of the quantities $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$: and in α_{m+s}, the expressions, of which the initial constants $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are coefficients, are polynomials of degree $s+1$ in the coefficients of the functions $P_1(z-\zeta), \ldots, P_m(z-\zeta)$. The earlier investigation shews that the power-series for w converges; accordingly, the determination of the coefficients α in this manner leads to the formal expression of an integral w satisfying the equation.

5. Further, the integral thus obtained is the only regular function, which is a solution of the equation and satisfies the initial conditions associated with $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. If it were possible to have any other regular function, which also is a solution and satisfies the same initial conditions, its expression would be of the form

$$w' = \sum_{\mu=0}^{m-1} \frac{\alpha_\mu}{\mu!}(z-\zeta)^\mu + \sum_{\mu=m}^{\infty} \frac{\alpha'_\mu}{\mu!}(z-\zeta)^\mu,$$

a regular function of $z - \zeta$. The coefficients would be determinable, as before, from a relation

$$\frac{\alpha'_{m+s}}{s\,!} = A'_{m+s},$$

where A'_{m+s} is the same function of $\alpha_0, \ldots, \alpha_{m-1}, \alpha'_m, \ldots, \alpha'_{m+s-1}$ as A_{m+s} is of $\alpha_0, \ldots, \alpha_{m-1}, \alpha_m, \ldots, \alpha_{m+s-1}$. Hence

$$\alpha'_m = A'_m \quad = A_m = \alpha_m;$$

$$\alpha'_{m+1} = A'_{m+1} = A_{m+1}, \text{ after substitution for } \alpha'_m,$$

$$= \alpha_{m+1};$$

and so on, in succession. The coefficients agree, and the two series are the same, so that $w = w'$; and therefore the initial conditions uniquely determine an integral of the equation, which is a regular function of $z - \zeta$ in the domain of the ordinary point ζ.

COROLLARY I. *If all the initial constants* $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ *are zero, then the synectic integral of the equation is identically zero.* For in the preceding discussion it has been proved that α_{m+s}, for all the values of s, is a linear homogeneous function of $\alpha_0, \ldots, \alpha_{m-1}$; hence, in the circumstances contemplated, $\alpha_{m+s} = 0$ for all the values of s. Thus every coefficient in the series vanishes; accordingly, the integral is an identical zero.

COROLLARY II. *The initial constants* $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ *occur linearly in the expression of the synectic integral; and each of the* m *variable quantities, which have those constants for coefficients, is a synectic integral of the equation.* The first part is evident, because all the coefficients in w are linear and homogeneous in $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. As regards the second part, the variable quantity multiplied by α_s is derivable from w by making $\alpha_s = 1$, and all the other constants α equal to zero; these constitute a particular set of initial values which, according to the theorem, determine a synectic integral of the equation. Thus the synectic integral, determined by the initial values $\alpha_0, \ldots, \alpha_{m-1}$, is of the form

$$\alpha_0 u_1 + \alpha_1 u_2 + \ldots + \alpha_{m-1} u_m,$$

where each of the quantities $u_1, u_2, \ldots, u_m$ is a synectic integral of the equation.

Note 1. The series of powers of $z - \zeta$, which represents the synectic integral, has been proved to converge within the domain of ζ, so that its radius of convergence is $|a-\zeta|$, where a is the singularity of the coefficients which is nearest to ζ. All these singularities lying in the finite part of the plane are determinable by mere inspection of the forms of the coefficients: another method must be adopted in order to take account of a possible singularity when $z=\infty$ because, even though $z=\infty$ may be an ordinary point of the coefficients, infinite values of the variable affect the character of w and its derivatives.

For this purpose, we may change the variable by the substitution

$$zx = 1,$$

and we then consider the relation of the x-origin to the transformed equation as a possible singularity. The transformation of the equation is immediately obtained by means of the formula

$$\frac{d^k w}{dz^k} = (-1)^k \sum_{\alpha=1}^{k} \frac{k!\,(k-1)!}{\alpha!\,(\alpha-1)!} \frac{x^{k+\alpha}}{(k-\alpha)!} \frac{d^\alpha w}{dx^\alpha};$$

inspection of the transformed equation then shews whether $x=0$ is, or is not, a singularity. Or, without changing the independent variable, we may consider a series for w in descending powers of z: examples will occur hereafter.

It may happen that there is no singularity of the coefficients in the finite part of the plane, infinite values then providing the only singularity. In that case, we should not take the quantity r in the preceding investigation as equal to $|\infty-\zeta|$, that is, as infinite; it would suffice that r should be finite, though as large as we please.

It may happen that there is no singularity of the coefficients for either finite or infinite values of z; if the coefficients are uniform, they then can only be constants. The dominant equation is then effectively the same as the original equation; the investigation is still applicable, but it furnishes less information as to the result than a method which will be indicated later (§ 6).

Note 2. The preceding proof is based upon that which is given* by Fuchs in his initial, and now classical, memoir on the theory of linear differential equations.

* *Crelle*, t. LXVI (1866), pp. 122—125.