

## Part I

# GEOPHYSICAL FLUIDS

## CHAPTER

## 1

# Fluid Fundamentals

**F**LUIDS, LIKE SOLIDS, move if they are pushed and they warm if they are heated. But, unlike solids, they flow and deform. In this chapter we establish the governing equations of motion for a fluid, with particular attention to air and seawater — the fluids of the atmosphere and ocean, respectively. Readers who already have knowledge of fluid dynamics may skim this chapter and begin reading more seriously at Chapter 2, where we begin to look at the effects of rotation and stratification.

## 1.1 TIME DERIVATIVES FOR FLUIDS

### 1.1.1 Field and Material Viewpoints

In solid-body mechanics one is normally concerned with the position and momentum of an identifiable object, such as a football or a planet, as it moves through space. In principle we could treat fluids the same way and try to follow the properties of individual fluid parcels as they flow along, perhaps getting hotter or colder as they move. This perspective is known as the *material* or *Lagrangian* viewpoint. However, in fluid dynamical problems we generally would like to know what the values of velocity, density and so on are at *fixed points* in space as time passes. A weather forecast we care about tells us how warm it will be where we live and, if we are given that, we may not care where a particular fluid parcel comes from or where it subsequently goes. Since the fluid is a continuum, this knowledge is equivalent to knowing how the fields of the dynamical variables evolve in space and time. This viewpoint is known as the *field* or *Eulerian* viewpoint.

Although the field viewpoint will often turn out to be the most practically useful, the material description is invaluable both in deriving the equations and in the subsequent insight it frequently provides. This is because the important quantities from a fundamental point of view are

The fluid dynamical equations of motion determine the evolution of a fluid. The equations are based on Newton's laws of motion and the laws of thermodynamics, and embody the principles of conservation of momentum, energy and mass. Initial conditions and boundary conditions are needed to solve the equations.

*The Lagrangian viewpoint is named for the Franco-Italian J. L. Lagrange (1736–1813), one of the most renowned mathematicians of his time. The Eulerian point of view is named for Leonhard Euler (1707–1783), the great Swiss mathematician. In fact, Euler is also largely responsible for the Lagrangian view, but the attribution became tangled over time.*

often those which are associated with a given fluid element: it is these which directly enter Newton's laws of motion and the thermodynamic equations. It is thus important to have a relationship between the rate of change of quantities associated with a given fluid element and the local rate of change of a field. The material derivative (also called the advective derivative or Lagrangian derivative) provides this relationship.

### 1.1.2 The Material Derivative of a Fluid Property

A *fluid element* is an infinitesimal, indivisible, piece of fluid — effectively a very small fluid parcel of fixed mass. The *material derivative*, or the *Lagrangian derivative*, is the rate of change of a property (such as temperature or momentum) of a particular fluid element or finite mass of fluid; that is, it is the total time derivative of a property of a piece of fluid.

Let us suppose that a fluid is characterized by a given velocity field  $\mathbf{v}(\mathbf{x}, t)$ , which determines its velocity throughout. Let us also suppose that the fluid has another property  $\varphi$ , and let us seek an expression for the rate of change of  $\varphi$  of a fluid element. Since  $\varphi$  is changing in time and in space we use the chain rule,

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z = \frac{\partial\varphi}{\partial t}\delta t + \delta\mathbf{x} \cdot \nabla\varphi. \quad (1.1)$$

This is true in general for any  $\delta t$ ,  $\delta x$ , etc. The total time derivative is then

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\varphi. \quad (1.2)$$

If this equation is to provide a material derivative we must identify the time derivative in the second term on the right-hand side with the rate of change of position of a fluid element, namely its velocity. Hence, the material derivative of the property  $\varphi$  is

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi. \quad (1.3)$$

The right-hand side expresses the material derivative in terms of the local rate of change of  $\varphi$  plus a contribution arising from the spatial variation of  $\varphi$ , experienced only as the fluid parcel moves. Because the material derivative is so common, and to distinguish it from other derivatives, we denote it by the operator  $D/Dt$ . Thus, the material derivative of the field  $\varphi$  is

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + (\mathbf{v} \cdot \nabla)\varphi. \quad (1.4)$$

The brackets in the last term of this equation are helpful in reminding us that  $(\mathbf{v} \cdot \nabla)$  is an operator acting on  $\varphi$ . The operator  $\partial/\partial t + (\mathbf{v} \cdot \nabla)$  is the *Eulerian representation of the Lagrangian derivative as applied to a field*.

### Material derivative of vector field

The material derivative may act on a vector field  $\mathbf{b}$ , in which case

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (1.5)$$

In Cartesian coordinates this is

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + u\frac{\partial\mathbf{b}}{\partial x} + v\frac{\partial\mathbf{b}}{\partial y} + w\frac{\partial\mathbf{b}}{\partial z}, \quad (1.6)$$

and for a particular component of  $\mathbf{b}$ ,  $b^x$  say,

$$\frac{Db^x}{Dt} = \frac{\partial b^x}{\partial t} + u\frac{\partial b^x}{\partial x} + v\frac{\partial b^x}{\partial y} + w\frac{\partial b^x}{\partial z}, \quad (1.7)$$

and similarly for  $b^y$  and  $b^z$ . In coordinate systems other than Cartesian the advective derivative of a vector is not simply the sum of the advective derivatives of its components, because the coordinate vectors themselves change direction with position; this will be important when we deal with spherical coordinates.

### 1.1.3 Material Derivative of a Volume

The volume that a given, unchanging, mass of fluid occupies is deformed and advected by the fluid motion, and there is no reason why it should remain constant. Rather, the volume will change as a result of the movement of each element of its bounding material surface, and in particular it will change if there is a non-zero normal component of the velocity at the fluid surface. That is, if the volume of some fluid is  $\int dV$ , then

$$\frac{D}{Dt} \int_V dV = \int_S \mathbf{v} \cdot d\mathbf{S}, \quad (1.8)$$

where the subscript  $V$  indicates that the integral is a definite integral over some finite volume  $V$ , and the limits of the integral are functions of time since the volume is changing. The integral on the right-hand side is over the closed surface,  $S$ , bounding the volume. Although intuitively apparent (to some), this expression may be derived more formally using Leibniz's formula for the rate of change of an integral whose limits are changing. Using the divergence theorem on the right-hand side, (1.8) becomes

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} dV. \quad (1.9)$$

The rate of change of the volume of an infinitesimal fluid element of volume  $\Delta V$  is obtained by taking the limit of this expression as the volume tends to zero, giving

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \frac{D\Delta V}{Dt} = \nabla \cdot \mathbf{v}. \quad (1.10)$$

The Eulerian derivative is the rate of change of a property at a fixed location in space. The material derivative is the rate of change of a property of a given piece of fluid, which may be moving and so changing its position.

We will often write such expressions informally as

$$\frac{D\Delta V}{Dt} = \Delta V \nabla \cdot \mathbf{v}, \tag{1.11}$$

with the limit implied.

Consider now the material derivative of some fluid property,  $\xi$  say, multiplied by the volume of a fluid element,  $\Delta V$ . Such a derivative arises when  $\xi$  is the amount per unit volume of  $\xi$ -substance — the mass density or the amount of a dye per unit volume, for example. Then we have

$$\frac{D}{Dt}(\xi \Delta V) = \xi \frac{D\Delta V}{Dt} + \Delta V \frac{D\xi}{Dt}. \tag{1.12}$$

Using (1.11) this becomes

$$\frac{D}{Dt}(\xi \Delta V) = \Delta V \left( \xi \nabla \cdot \mathbf{v} + \frac{D\xi}{Dt} \right), \tag{1.13}$$

and the analogous result for a finite fluid volume is just

$$\frac{D}{Dt} \int_V \xi \, dV = \int_V \left( \xi \nabla \cdot \mathbf{v} + \frac{D\xi}{Dt} \right) dV. \tag{1.14}$$

This expression is to be contrasted with the Eulerian derivative for which the volume, and so the limits of integration, are fixed and we have

$$\frac{d}{dt} \int_V \xi \, dV = \int_V \frac{\partial \xi}{\partial t} \, dV. \tag{1.15}$$

Now consider the material derivative of a fluid property  $\varphi$  multiplied by the mass of a fluid element,  $\rho \Delta V$ , where  $\rho$  is the fluid density. Such a derivative arises when  $\varphi$  is the amount of  $\varphi$ -substance per unit mass (note, for example, that the momentum of a fluid element is  $\rho \mathbf{v} \Delta V$ ). The material derivative of  $\varphi \rho \Delta V$  is given by

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D\varphi}{Dt} + \varphi \frac{D}{Dt}(\rho \Delta V). \tag{1.16}$$

But  $\rho \Delta V$  is just the mass of the fluid element, and that is constant — that is how a fluid element is defined. Thus the second term on the right-hand side vanishes and

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D\varphi}{Dt} \quad \text{and} \quad \frac{D}{Dt} \int_V \varphi \rho \, dV = \int_V \rho \frac{D\varphi}{Dt} \, dV, \tag{1.17a,b}$$

where (1.17b) applies to a finite volume. That expression may also be derived more formally using Leibniz’s formula for the material derivative of an integral, and the result also holds when  $\varphi$  is a vector. The result is quite different from the corresponding Eulerian derivative, in which the volume is kept fixed; in that case we have:

$$\frac{d}{dt} \int_V \varphi \rho \, dV = \int_V \frac{\partial}{\partial t}(\varphi \rho) \, dV. \tag{1.18}$$

Various material and Eulerian derivatives are summarized in the shaded box on the facing page.

### Material and Eulerian Derivatives

The material derivatives of a scalar ( $\varphi$ ) and a vector ( $\mathbf{b}$ ) field are given by:

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi, \quad \frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (\text{D.1})$$

Various material derivatives of integrals are:

$$\frac{D}{Dt} \int_V \varphi \, dV = \int_V \left( \frac{D\varphi}{Dt} + \varphi \nabla \cdot \mathbf{v} \right) dV = \int_V \left( \frac{\partial\varphi}{\partial t} + \nabla \cdot (\varphi\mathbf{v}) \right) dV, \quad (\text{D.2})$$

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} \, dV, \quad (\text{D.3})$$

$$\frac{D}{Dt} \int_V \rho\varphi \, dV = \int_V \rho \frac{D\varphi}{Dt} \, dV. \quad (\text{D.4})$$

These formulae also hold if  $\varphi$  is a vector. The Eulerian derivative of an integral is:

$$\frac{d}{dt} \int_V \varphi \, dV = \int_V \frac{\partial\varphi}{\partial t} \, dV, \quad (\text{D.5})$$

so that

$$\frac{d}{dt} \int_V dV = 0 \quad \text{and} \quad \frac{d}{dt} \int_V \rho\varphi \, dV = \int_V \frac{\partial\rho\varphi}{\partial t} \, dV. \quad (\text{D.6})$$

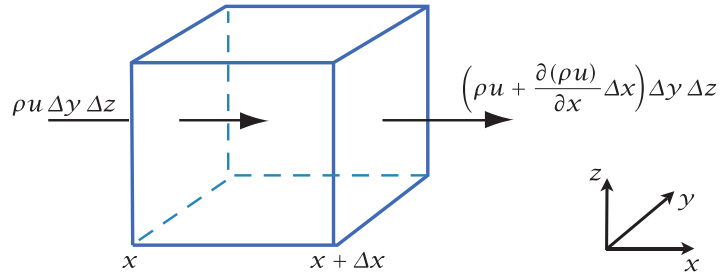
## 1.2 THE MASS CONTINUITY EQUATION

In classical mechanics mass is absolutely conserved and in solid-body mechanics we normally do not need an explicit equation of mass conservation. However, in fluid mechanics a fluid may flow into and away from a particular location, and fluid density may change, and we need an equation to describe that change.

### 1.2.1 An Eulerian Derivation

We first derive the mass conservation equation from an Eulerian point of view; that is, our reference frame is fixed in space and the fluid flows through it. Consider an infinitesimal, rectangular cuboid, control volume,  $\Delta V = \Delta x \Delta y \Delta z$  that is fixed in space, as in Fig. 1.1. Fluid moves into or out of the volume through its surface, including through its faces in the  $y$ - $z$  plane of area  $\Delta A = \Delta y \Delta z$  at coordinates  $x$  and  $x + \Delta x$ . The accumulation of fluid within the control volume due to motion in the  $x$ -direction is

**Fig. 1.1:** Mass conservation in an Eulerian cuboid control volume. The mass convergence,  $-\partial(\rho u)/\partial x$  (plus contributions from the  $y$  and  $z$  directions), must be balanced by a density increase equal to  $\partial\rho/\partial t$ .



evidently

$$\Delta y \Delta z [(\rho u)(x, y, z) - (\rho u)(x + \Delta x, y, z)] = -\left. \frac{\partial(\rho u)}{\partial x} \right|_{x,y,z} \Delta x \Delta y \Delta z. \quad (1.19)$$

To this must be added the effects of motion in the  $y$ - and  $z$ -directions, namely

$$-\left[ \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \Delta x \Delta y \Delta z. \quad (1.20)$$

This net accumulation of fluid must be accompanied by a corresponding increase of fluid mass within the control volume. This is

$$\frac{\partial}{\partial t} (\text{density} \times \text{volume}) = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}, \quad (1.21)$$

because the volume is constant. Thus, because mass is conserved, (1.19), (1.20) and (1.21) give

$$\Delta x \Delta y \Delta z \left[ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = 0. \quad (1.22)$$

The quantity in square brackets must be zero and we therefore have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.23)$$

This is called the *mass continuity equation* for it recognizes the continuous nature of the mass field in a fluid. There is no diffusion term in (1.23), no term like  $\kappa \nabla^2 \rho$ . This is because mass is transported by the macroscopic movement of molecules; even if this motion appears diffusion-like, any net macroscopic molecular motion constitutes, by definition, a velocity field.

Neither (1.23) nor the derivation that leads to it depends in any way on Cartesian geometry; a more general vector derivation using an arbitrary control volume is left as an easy exercise for the reader.

### 1.2.2 Mass Continuity via the Material Derivative

We now derive the mass continuity equation (1.23) from a material perspective. This is the most fundamental approach of all since the principle of mass conservation states simply that the mass of a given element

## 1.2 THE MASS CONTINUITY EQUATION

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of fluid is, by definition of the element, constant. Thus, consider a small mass of fluid of density  $\rho$  and volume  $\Delta V$ . Then conservation of mass may be represented by

$$\frac{D}{Dt}(\rho\Delta V) = 0. \quad (1.24)$$

Both the density and the volume of the parcel may change, so

$$\Delta V \frac{D\rho}{Dt} + \rho \frac{D\Delta V}{Dt} = \Delta V \left( \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right) = 0, \quad (1.25)$$

where the second expression follows using (1.11). Since the volume element is non-zero the term in brackets must vanish and

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (1.26)$$

After expansion of the first term this becomes identical to (1.23). (A slightly more formal way to derive this result uses (1.14) with  $\xi$  replaced by  $\rho$ .) Summarizing, equivalent partial differential equations representing conservation of mass are

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.27a,b)$$

### 1.2.3 Incompressible Fluids

A near-universal property of liquids is that their density is nearly constant; that is, they are essentially *incompressible*. If we write the density as

$$\rho(x, y, z, t) = \rho_0 + \delta\rho(x, y, z, t), \quad (1.28)$$

where  $\rho_0$  is a constant, then a truly incompressible fluid has  $\delta\rho = 0$ . No fluid is incompressible in this strict sense so we relax the meaning slightly and simply require  $|\delta\rho| \ll \rho_0$ . When this is satisfied the mass continuity equation, (1.27a) takes on a different form. Equation (1.27a) may be written, without approximation, as

$$\frac{D\delta\rho}{Dt} + (\rho_0 + \delta\rho)\nabla \cdot \mathbf{v} = 0. \quad (1.29)$$

If the fluid is incompressible then the terms involving  $\delta\rho$  are much smaller than those involving  $\rho_0$  and hence may be neglected, giving

$$\nabla \cdot \mathbf{v} = 0. \quad (1.30)$$

This is the mass continuity equation for an incompressible fluid, and its satisfaction may be taken as the defining quality of an incompressible fluid. The *prognostic* equation, (1.27) has become a *diagnostic* equation.

An incompressible fluid is sometimes defined as one whose density is not affected by pressure. This definition may usefully be generalized to mean a fluid whose density is very nearly constant (and so also not affected by temperature or composition) such that the mass continuity equation takes the form (1.30). That equation is normally a very good approximation for seawater, less so for air.



*The viscous form of the fluid dynamical equations of motion was established by Claude-Louis-Marie-Henri Navier (1785–1836) a French civil engineer, and George Stokes (1819–1903), an Anglo-Irish applied mathematician, who further elucidated viscous effects. Prior to their work the great Swiss mathematician Leonard Euler (1707–1783) had established the general form of the fluid equations for an inviscid incompressible flow, namely the Euler equations.*

### 1.3 THE MOMENTUM EQUATION

The momentum equation is a partial differential equation that describes how the velocity or momentum of a fluid responds to internal and imposed forces. We derive it here using material methods, with a very heuristic treatment of the terms representing pressure and viscous forces.

#### 1.3.1 Advection

Let  $\mathbf{m}(x, y, z, t)$  be the momentum-density field (momentum per unit volume) of the fluid. Thus,  $\mathbf{m} = \rho\mathbf{v}$  and the total momentum of a volume of fluid is given by the volume integral  $\int_V \mathbf{m} dV$ . Now, for a fluid the rate of change of momentum of an identifiable fluid mass is given by the material derivative, and by Newton's second law this is equal to the force acting on it. Thus,

$$\frac{D}{Dt} \int_V \rho\mathbf{v} dV = \int_V \mathbf{F} dV, \tag{1.31}$$

where  $\mathbf{F}$  is the force per unit volume. Now, using (1.17b) (with  $\varphi$  replaced by  $\mathbf{v}$ ) to transform the left-hand side of (1.31), we obtain

$$\int_V \left( \rho \frac{D\mathbf{v}}{Dt} - \mathbf{F} \right) dV = 0. \tag{1.32}$$

Because the volume is arbitrary the integrand itself must vanish and we obtain

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F} \quad \text{or} \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\mathbf{F}}{\rho}, \tag{1.33a,b}$$

having used (1.5) to expand the material derivative. We have thus obtained an expression for how a fluid accelerates if subject to known forces. As well as external forces (like gravity), a stress arises from the direct contact between one fluid parcel and another, giving rise to pressure and viscous forces, sometimes referred to as *contact forces*.

#### 1.3.2 Pressure and Viscous Forces

##### Pressure

Within or at the boundary of a fluid the pressure is the normal force per unit area due to the collective action of molecular motion. Thus

$$d\hat{\mathbf{F}}_p = -p d\mathbf{S}, \tag{1.34}$$

where  $p$  is the pressure,  $\hat{\mathbf{F}}_p$  is the pressure force and  $d\mathbf{S}$  an infinitesimal surface element. If we grant ourselves this intuitive notion, it is a simple matter to assess the influence of pressure on a fluid, for the pressure force on a volume of fluid is the integral of the pressure over the its boundary and so

$$\hat{\mathbf{F}}_p = - \int_S p d\mathbf{S}. \tag{1.35}$$

The forces due to pressure and viscosity are 'contact forces' arising because of the inter-molecular forces and/or collisions in a fluid. The net pressure force on a fluid element is proportional to the gradient of pressure.

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The minus sign arises because the pressure force is directed inwards, whereas  $\mathbf{S}$  is a vector normal to the surface and directed outwards. Applying a form of the divergence theorem to the right-hand side gives

$$\hat{\mathbf{F}}_p = - \int_V \nabla p \, dV, \quad (1.36)$$

where the volume  $V$  is bounded by the surface  $S$ . The pressure gradient force per unit volume,  $\mathbf{F}_p$ , is therefore just  $-\nabla p$ .

#### Viscosity

The effects of viscosity are apparent in many situations — the flow of tectonic plates or volcanic lava are obvious examples. The viscous force per unit volume is approximately equal to  $\mu \nabla^2 \mathbf{v}$ , where  $\mu$  is the coefficient of viscosity. With the pressure and viscous terms the momentum equation becomes,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{F}_b, \quad (1.37)$$

where  $\nu = \mu/\rho$  is the *kinematic viscosity* and  $\mathbf{F}_b$  represents body forces (per unit mass) such as gravity,  $\mathbf{g}$ . For most large-scale flows in the atmosphere and ocean the viscous term is, in fact, negligibly small.

#### 1.3.3 The Hydrostatic Approximation

Neglecting viscosity, the vertical component (the component parallel to the gravitational force,  $\mathbf{g}$ ) of the momentum equation is

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \quad (1.38)$$

where  $w$  is the vertical component of the velocity and  $\mathbf{g} = -g\hat{\mathbf{k}}$ . If the fluid is static the gravitational term is balanced by the pressure term and we have

$$\frac{\partial p}{\partial z} = -\rho g, \quad (1.39)$$

and this relation is known as *hydrostatic balance*, or hydrostasy. It is clear in this case that the pressure at a point is given by the weight of the fluid above it, provided that  $p = 0$  at the top of the fluid. The flow need not be static for hydrostasy to hold — equation (1.39) is a good *approximation* to (1.38) provided that the vertical acceleration,  $Dw/Dt$ , is sufficiently small compared to gravity, which is nearly always the case in both atmosphere and ocean except in intense storms. However, because the pressure also appears in the horizontal momentum equations, hydrostatic balance must be *very* well satisfied to ensure that (1.39) provides an accurate enough pressure to determine the horizontal pressure gradients, a point we return to in Section 3.2.

*Equation (1.37) is sometimes called the Navier–Stokes equation. If viscosity is absent the equation is the Euler equation. Sometimes these names are taken as applying to the complete set of equations of motion.*

*Hydrostatic balance is an approximation to the vertical momentum equation, valid for large-scale motion in both atmosphere and ocean and normally very well satisfied for flows of horizontal scales greater than a few tens of kilometres. It is one of the most fundamental and useful approximations in atmospheric and oceanic dynamics.*