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DIFFERENTIAL EQUATIONS.

PART IV.  
PARTIAL DIFFERENTIAL EQUATIONS.

BY  
ANDREW RUSSELL FORSYTH,  
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