CHAPTER XII.

GENERAL INTEGRALS OF EQUATIONS OF ORDERS HIGHER THAN THE FIRST.

THE present chapter is devoted to general explanations, connected with the existence-theorem and with the kinds of integrals that are possessed by equations of order higher than the first, particularly those equations having general integrals without partial quadratures. For the most part, though not entirely, the equations considered are of the second order in two independent variables. The discussion is based mainly upon the memoir of Ampère, quoted in § 179, and upon the first chapter of the memoir by Imschenetsky, quoted in § 180.

178. After the discussion of equations of the first order which involve only a single dependent variable, and a discussion of sets of equations of the first order which involve several dependent variables and are integrable by any generalisation of any process that is effective for equations involving only one dependent variable, the next subject for consideration is manifestly the theory of partial differential equations of the second order. We shall begin with the simplest aggregate of such equations; and, for that aggregate, we shall assume that there is only a single dependent variable z, and that there are only two independent variables x and y. Denoting the first derivatives of z by p and q, and the second derivatives by r, s and t, as usual, we may take an equation of the second order in the form

$$f(x, y, z, p, q, r, s, t) = 0.$$

Such an equation certainly possesses integrals. We shall assume that f either is in a form or can be brought into a form which makes it a regular function of its arguments: in that case, we have seen that Cauchy's existence-theorem applies and that integrals, characterised by certain properties, do exist. Thus there is an integral z determined by the characteristic properties:—

(i) it is a regular function of x and y within fields of variation round a and b, given by

$$|x-a| \leqslant \rho, \quad |y-b| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) when $x=a$, the integral z reduces to $\phi_0(y)$ and the derivative $\frac{\partial z}{\partial x}$ reduces to $\phi_1(y)$, where $\phi_0(y)$ and $\phi_1(y)$ are regular functions of y within the domain of b and are otherwise arbitrary.

There is a single condition, of a formal type, which must be satisfied, or the existence of the foregoing integral cannot be established: it is that, if

$$\phi_0(b)=c, \quad \phi_1(b)=\lambda, \quad \phi_0'(b)=\mu, \quad \phi_1'(b)=\beta, \quad \phi_0''(b)=\gamma,$$

the equation

$$f(a, b, c, \lambda, \mu, \theta, \beta, \gamma)=0,$$

regarded as an equation in θ, should have at least one simple root. When this condition is satisfied, each simple root θ determines an integral z as above: and the integral thus associated with that simple root is unique.

When the condition is not satisfied so that $f=0$, regarded as an equation in θ, has no simple root, then the existence of such an integral is not established. But it may then happen that $f=0$, regarded as an equation in γ, has simple roots. In that case, the theorem establishes the existence of an integral z, regular as before in the domain of a and b, but now such that, when $y=b$, the integral z reduces to $\psi_0(x)$ and the derivative $\frac{\partial z}{\partial y}$ reduces to $\psi_1(x)$, where $\psi_0(x)$ and $\psi_1(x)$ are regular functions of x within the domain of a and are otherwise arbitrary.

If, however, $f=0$, regarded as an equation in γ, has no simple roots and, as before, has no simple roots when regarded as an equation in θ, it may have simple roots when regarded as an equation in β. In that case, there is an integral of a similar type, obtainable most easily through a transformation of the independent variables.

Thus the equation is proved to possess an integral characterised by definite properties except only in the case where $f=0$, regarded as an equation in θ, β, γ in turn, possesses no simple roots, so that we should have

$$\frac{\partial f}{\partial \theta}=0, \quad \frac{\partial f}{\partial \beta}=0, \quad \frac{\partial f}{\partial \gamma}=0,$$

concurrently with $f=0$. Returning now to the differential equation

$$f(x, y, z, p, q, r, s, t)=0,$$

the quantities

$$\frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial s}, \quad \frac{\partial f}{\partial t}$$

will usually be variable quantities; and then values can usually be given to their arguments such that, while f vanishes for those values, not all the three quantities $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ vanish. It may, however, happen that there are values of the arguments which satisfy the four equations

$$f=0, \quad \frac{\partial f}{\partial r}=0, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=0;$$

and then the Cauchy existence-theorem does not apply. In that case, there may be variable values of z (and of x and y) which satisfy all the four equations: such values of z, if any, will be called *singular integrals*. In all other cases, the existence-theorem establishes the existence of an integral with the specified properties: as its existence was first established by Cauchy, it frequently is called the *Cauchy integral*.

It will be noticed that two arbitrary functions enter into the expression of the specified properties.

The form thus stated is the simplest form of the Cauchy integral, in so far that the initial conditions are in their simplest form. As indicated (§ 24) in the discussion of the existence theorem, the initial conditions can be taken in an ampler form as follows:

> For all the values of x and y satisfying a given relation that is not critical with regard to the form of the differential equation, that is, for all points of a given analytical plane

curve, the variable z and its derivative in any direction, that is not tangential to the curve, acquire values represented by arbitrarily assigned functions of x and y.

This undoubtedly is more general. However, as it arises through a transformation of the variables from the simpler case, and does not otherwise add any element of generality to the solution, we shall usually be content to take the initial conditions in their simpler form.

It will be convenient to assume, for the purpose of immediate discussion as well as for simplicity of statement, that the equation can be resolved with regard to r, so that it takes the form

$$r = g(x, y, z, p, q, s, t):$$

the original equation can be regarded as the aggregate of all these resolved equations. The Cauchy integral is then a regular function of the variables in the domain of a and b: two arbitrary functions $\phi_0(y)$ and $\phi_1(y)$, subject solely to the condition of being regular in the domain of b, affect its form: and it is a unique integral as satisfying these conditions.

But while it thus possesses arbitrarily assigned elements, which frequently can be specialised so as to include integrals otherwise obtained, there is no certainty that specialisation or definition of these elements will secure that the integral shall include every integral; and therefore there is no certainty that the Cauchy integral is completely comprehensive. A question thus arises as to whether the equation possesses any integral that is more comprehensive; a further question is stirred as to the different kinds of integral that the equation may possess. Even so, limiting assumptions have been made: all singularities and other deviations from regularity in the form of the original equation have been avoided.

Two Definitions of the General Integral.

179. It is usual to assign, to the most comprehensive integral known, the name of the *general integral*, for partial equations of order higher than the first; but there are two definitions of the general integral. One of these definitions is due* to Ampère;

* *Journn. de l'Éc. Polytechnique, cah.* XVII (1815), p. 550.

the other, given* by Darboux, is based upon the researches of Cauchy.

According to Ampère, an integral (no matter how obtained) is general when the only relations, which are free from the arbitrary elements and to which the integral leads among the variables and the derivatives of the dependent variable up to any order whatever, are those expressed by the differential equation and by equations deduced from the equation by differentiation. Thus, in this sense,

$$z = \phi(x+y) + \psi(x-y)$$

is a general integral of the equation

$$r = t;$$

for the relations to which the integral equation gives rise are

$$\frac{\partial^{2+m+n} z}{\partial x^{2+m}\,\partial y^n} = \frac{\partial^{2+m+n} z}{\partial x^m \partial y^{2+n}},$$

for all positive integer values of m and n, and all of these relations are derivatives of the differential equation. But

$$z = a(x^2 + y^2) + 2hxy + bx + cy + d,$$

where a, b, c, d, h are arbitrary constants, is not a general integral of the same equation: for it satisfies relations

$$\frac{\partial^3 z}{\partial x^3} = 0, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 z}{\partial y^3} = 0,$$

no one of which can be derived from the differential equation, though derivatives of the differential equation are not inconsistent with the relations.

According to Darboux, an integral (no matter how obtained) is general if the arbitrary elements which it contains can be determined so as to give the Cauchy integral, involving assigned functional values to z and to one of its derivatives in specified circumstances. Thus, in this sense also,

$$z = \phi(x+y) + \psi(x-y)$$

is a general integral of the equation

$$r = t:$$

* *Théorie générale des surfaces*, t. II, pp. 97, 98.

for, if the initial conditions of the Cauchy integral are that $z = f(y)$ and $\dfrac{\partial z}{\partial x} = g(y)$, when $x = a$, then if

$$\sigma(u) = \tfrac{1}{2} f(u) + \tfrac{1}{2} \int_0^u g(u)\, du,$$

$$\rho(v) = \tfrac{1}{2} f(v) - \tfrac{1}{2} \int_0^v g(v)\, dv,$$

the required Cauchy integral is given by

$$z = \sigma(x + y - a) + \rho(a - x + y),$$

the functions ϕ and ψ thus having been appropriately determined.

The tenour of the Ampère definition of a general integral is different from that of the Cauchy general integral. Though the difference between the integrals is often of no account, yet for particular equations it can be significant: and therefore it is worth while to estimate which of the two integrals is the more comprehensive.

It seems clear that, in the matter of comprehensiveness, the Cauchy general integral has some advantage over the Ampère general integral.

The limitations, which are imposed in the course of establishing the Cauchy integral, are of a qualitative kind: they are restrictions as to the position and the extent of the domains within which the various functions that occur are regular, or they are hypotheses as to the resolubility of the differential equation: but no positive relations (other than derivatives of the differential equation) are used or are required in order to secure the convergence of the power-series obtained, or the continuity of the functions, or the freedom of the initial conditions. Consequently, an integral that is general in the sense of the Darboux-Cauchy definition is general also in the sense of Ampère's definition. As against this inference, it must be borne in mind that, however arbitrarily the initial conditions are chosen either as regards the position of the domain or the forms of the assigned functions, the Cauchy integral is always a regular function of the variables and that deviations from regularity have been excluded from consideration: there is no such restriction on the Ampère integral.

The restriction can often be partly removed by considering a part of a domain and by taking a regular expression for a branch of a non-regular integral in that region: but this modification is not always possible, and there are deviations from regularity such that the removal of the restriction cannot be made complete.

But on the other hand, classes of equations can be constructed which have integrals that are general in the Ampère sense and certainly are not general in the Darboux-Cauchy sense. It is true that such classes of equations are special in type, and that a similar difference need not exist for equations that are not of any special type: but the mere existence of such equations is a limitation upon the comparative comprehensiveness of the Ampère integral.

An instance is adduced* by Goursat in the example

$$s = yq:$$

simple quadratures lead to an integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

where θ and ϕ are arbitrary functions, and a is any constant. Now the quantity z', where

$$z' = \int_a^y e^{xu} \phi(u)\, du,$$

satisfies the differential equation: all the relations between the variables and the derivatives of z', which are free from arbitrary elements, are constituted by the differential equation and by derivatives of the differential equation. Thus z' is an integral of the differential equation which is general in the sense of Ampère's definition: it clearly is less comprehensive than the integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

which is easily seen to be general in the Darboux-Cauchy sense.

We shall return to a further discussion of the matter when dealing with linear equations of the second order. It is manifest that the foregoing explanations can be applied to equations of

* *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II, p. 212. This treatise, when quoted hereafter, will be referred to as Goursat, *S. O.*

order higher than the second: and meanwhile, we may regard the general integral as one that is largely (though not universally) comprehensive.

Classes of Integrals.

180. Before proceeding to the discussion of certain properties of general integrals, taken according to either of the definitions just indicated, we shall mention some other classes of integrals and briefly outline some of their relations with one another*.

Speaking broadly, we may define an integral of a partial differential equation of the second (or of higher) order as a relation between the variables such that, in virtue of the relation itself and of derivatives from it, the differential equation is satisfied. When the integral relation does not involve derivatives of the dependent variable, it is sometimes called a *primitive*: the more frequent practice is not to give any special title to such a relation. When the integral relation does involve derivatives of the dependent variable, these derivatives being of order lower than that of the equation, the integral relation is usually called an *intermediate integral*: it has not been proved, and it is not a fact, that intermediate integrals are always possessed by differential equations of order higher than the first.

We have already referred to *general integrals*: after the provisional explanations and for the sake of simplicity, we shall regard the Ampère definition as giving a necessary qualification (though not a complete qualification) for a general integral. A *particular integral* is a special case of the general integral: it is distinguished by the property that while, in conjunction with its derivatives, it leads to the differential equation and to derivatives of the differential equation, it leads also to other differential equations not obtainable as derivatives of the differential equation.

It has been customary with writers, following Lagrange, to refer to *complete integrals* or *complete primitives*: Ampère however considered, and gave† reasons for considering, that such

* In this connection, reference may be made to the first chapter of Imschenetsky's memoir on equations of the second order with two independent variables, *Grunert's Archiv*, t. LIV (1872), pp. 209—360.

† See the memoir (p. 554) cited in § 179.

integrals are always particular integrals. Their special occurrence is due to the fact that Lagrange proceeded to a differential equation from an integral relation, by eliminating from the latter and from its derivatives, as many arbitrary elements as possible. Thus, let an integral equation

$$g(x, y, z, a_1, a_2, a_3, a_4, a_5) = 0$$

be given, involving five arbitrary constants. When the two first derivatives and the three second derivatives are formed, there are six equations in all: from these, the five constants can be eliminated and the eliminant may be a single equation of the second order. Also, if the constants be independent of one another, not more than a single equation will result, unless some functional combinations occur: and similarly, subject to the same exceptional occurrence of functional combinations, not more than five independent constants could be eliminated. Thus five is the greatest number of arbitrary constants which could be expected in an integral, when its generality depends on arbitrary constants alone; hence the name assigned* to such an integral by Lagrange.

There are other integrals of various types. We have seen that the equations

$$f = 0, \quad \frac{\partial f}{\partial r} = 0, \quad \frac{\partial f}{\partial s} = 0, \quad \frac{\partial f}{\partial t} = 0,$$

could be satisfied simultaneously; if they lead to an integral, on the analogy of the corresponding case for equations of the first order, it is called *singular*.

It is sufficiently clear that, just as in the case of equations of the first order where the more obvious classes of integrals are not sufficiently comprehensive to include all types of special integrals, so in the case of equations of higher order there will be integrals (which may be called *special*, for convenience), possessed by particular equations and not included in the preceding classes.

* A similar argument, applied to an equation

$$f(x, y, z, p, q, r, s, t) = 0,$$

shews that, if initial values chosen for x and y be regarded as pure constants, the values of the six quantities z, p, q, r, s, t for those initial values are connected by a single relation, so that it might be considered that there are five independent arbitrary constants at our disposal.

Frequently they will be peculiarly associated with the form of the equation when it is quite regular: when the equation is not regular, special integrals will frequently occur, particularly associated with singularities of the form or with other deviations from regularity*.

It is not unusual to attempt some classification of intermediate integrals, though this is the less important because such integrals do not always exist. Still, when they do exist, two classes are selected as being of wider range than others. If an intermediate integral involves one arbitrary function in its expression, it is usually called an *intermediate general integral*. If it involves two arbitrary independent constants (this being usually the greatest number of constants that can be eliminated from an integral and its two derivatives leading to an equation of the second order), it is sometimes called an *intermediate complete integral*. Well-known instances of equations possessing an intermediate general integral are provided by the equations

$$U(rt - s^2) + Rr + 2Ss + Tt = V,$$

when certain conditions are satisfied, the quantities R, S, T, U, V being functions of x, y, z, p, q. An instance of an equation possessing a complete intermediate integral is given by

$$(rt - s^2)^2 + (pt - qs)(ps - qr) = 0:$$

the intermediate integral is

$$c^2q - cz + p = a,$$

where a and c are arbitrary constants.

Instances hereafter will occur freely in which it appears that a differential equation of the second order does not possess any intermediate integral. Thus the equation†

$$s = z$$

cannot possess an intermediate integral. Such an integral, if possessed, would have one of the forms

$$p = f(x, y, z, q), \quad q = g(x, y, z), \quad p = h(x, y, z).$$

* See the Supplementary Note, at the end of Chapter XVI.

† The example is quoted by Imschenetsky, (*l.c.*) p. 222, from Raabe.