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A semi-stable case of the Shafarevich Conjecture

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Abstract

Suppose $K = W(k)[1/p]$, where $W(k)$ is the ring of Witt vectors with coefficients in an algebraically closed field k of characteristic $p \neq 2$. We discuss an explicit construction of p-adic semi-stable representations of the absolute Galois group of K with Hodge–Tate weights from $[0, p)$. This theory is applied to projective algebraic varieties over $\mathbb{Q}$ with good reduction outside 3 and semi-stable reduction modulo 3.

Introduction

In this expository paper we discuss the following result in the spirit of the Shafarevich Conjecture about non-existence of non-trivial abelian schemes over $\mathbb{Z}$.

Theorem 1.1. *If Y is a projective algebraic variety over $\mathbb{Q}$ with good reduction outside 3 and semi-stable reduction modulo 3 then $h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}})$.*

In particular, the above theorem implies that there are no such (non-trivial) abelian varieties Y (first proved in [13, 27]). Our result also eliminates a great deal of other varieties, e.g. all K3-surfaces.

The proof of Theorem 1.1 is given in [11] and is based on a:

– study of torsion subquotients of the Galois module $H^2_{et}(Y_{\bar{Q}}, \mathbb{Q}_3)$;
– modification of Breuil's torsion theory of semi-stable p-adic represenations with HT (Hodge–Tate) weights from $[0, p-1]$ over $W(k)$, where k is an algebraically closed field of characteristic p;

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– formalism of pre-abelian categories (short exact sequences, 6-term Hom – Ext exact sequences, p-divisible group objects, devissage);
– study of the group of fundamental units in $\mathbb{Q}(\sqrt[3]{3}, e^{2\pi i/9})$ (via the computing package SAGE).

The strategy of the proof is very close to the strategy used in the following "crystalline case" of the Shafarevich Conjecture [23, 7].

Theorem 1.2. *Suppose X is a projective algebraic variety over $\mathbb{Q}$ with everywhere good reduction. Then*

(a) $h^1(X_{\mathbb{C}}) = 0$, $h^2(X_{\mathbb{C}}) = h^{1,1}(X_{\mathbb{C}})$ *and* $h^3(X_{\mathbb{C}}) = 0$;
(b) $h^4(X_{\mathbb{C}}) = h^{2,2}(X_{\mathbb{C}})$ *under Generalized Riemann Hypothesis (GRH).*

Part (a) of this Theorem was obtained in [7] by studying the finite subquotients of the Galois modules $H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_5)$ with $1 \leqslant i \leqslant 3$. These Galois modules are unramified outside 5 and their local behaviour at 5 is described by the Fontaine–Laffaille theory [19] of p-adic torsion crystalline representations with HT weights from $[0, p-2]$. The approach in [7] is essentially similar to the approach from [23] but Fontaine considers etale cohomology with coefficients in $\mathbb{Q}_7$. (Of course, these results would be not possible without the great achievements of Fontaine's theory of p-adic periods.)

Part (b) was proved by the author in [7]. The proof requires the study of the Galois module $H^4_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_5)$, where the tools of the Fontaine–Laffaille theory are not sufficient. For this reason, we developed in [6] a modification of the Fontaine–Laffaille theory for crystalline representations with HT weights from $[0, p-1]$. Note that our modification of Breuil's theory works also in the context of crystalline representations and can be applied to reprove part b) of Theorem 1.2 (and similar results for varieties over $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$ from [7]). The appropriate comments will be given in due course below.

The constructions in [11] are very technical and we just sketch and discuss their basic steps. Most of them can be illustrated by earlier results related to the Shafarevich Conjecture, cf. Section 1.

In Sections 2–4 we work with a local field $K = \operatorname{Frac} W(k)$, where $W(k)$ is the ring of Witt vectors with coefficients in an algebracally closed field k of characteristic p, $p > 2$. Let $\bar{K}$ be an algebraic closure of K and $\Gamma_K = \operatorname{Gal}(\bar{K}/K)$. In Section 2 we outline the construction of the functor $\mathcal{V}^*$ from an appropriate category of filtered modules to the category of $\mathbb{F}_p[\Gamma_K]$-modules. This construction is based on the introduction of a modulo p "truncated" version of Fontaine's ring of p-adic semi-stable periods.

We associate to $\mathcal{V}^*$ the functor $\mathcal{CV}^*$ with values in the category of co-filtered $\mathbb{F}_p[\Gamma_K]$-modules and prove that this functor is fully faithful. In Section 3 we obtain the ramification estimates for the Galois modules H from the image of $\mathcal{V}^*$: if $v > 2 - 1/p$ then the higher ramification subgroups $\Gamma_K^{(v)}$ act trivially on H. We also obtain the ramification estimate for the Galois modules which are associated with the modulo p subquotients of crystalline representations with HT weights from $[0, p)$ and prove that both estimates are sharp. The methods we use here are close to the methods from [8, 9, 10]; one can use also our constructions to show that the estimates from [24] are sharp if $e = n = 1$. In Section 4 we explain the construction of our modification of Breuil's functor $\mathcal{V}^{ft}$. In fact, it is very close to the construction of the modification of the Fontaine–Laffaille functor from [6] but it can be developed in a simpler way due to advantages of Breuil's theory. One of the main features of this construction is that on the level of modulo p subquotients, $\mathcal{V}^{ft}$ essentially coincides with the functor $\mathcal{V}^*$ from Subsection 2. This gives the ramification estimates for modulo p subquotients of semi-stable and crystalline representations with HT weights from $[0, p)$. Finally, in Section 5 we outline the proofs of Theorems 1.1 and 1.2.

1. The Shafarevich Conjecture

Conjecture. (I. R. Shafarevich, 1962) *There are no projective algebraic curves over $\mathbb{Q}$ of genus $g \geqslant 1$ with everywhere good reduction, [29].*

The case $g = 1$ was considered by Shafarevich himself. He has just listed explicitly 22 elliptic curves over $\mathbb{Q}$ with good reduction outside 2 and verified that all these curves have bad reduction at 2. Later his PhD student (Volynsky) studied the case of curves of genus 2. This approach resulted in enormous calculations and was not published. In both cases the approach was based on the study of canonical equations for these curves. It became clear later that one should study the problem in a more general setting.

Conjecture. *There are no abelian varieties A over $\mathbb{Q}$ of dimension $g \geq 1$ with everywhere good reduction.*

This statement is easier to approach. The existence of such abelian variety would have provided examples of non-trivial p-divisible groups over $\mathbb{Z}$ (for all prime numbers p). The question about the existence of such p-divisible groups was asked by J.Tate in [31]. In this way the conjecture was proved in [21, 3] in 1985. The main features of used methods will be described below.

1.1. Small values of g

In [1, 2] it was proved that any 2-divisible group over $\mathbb{Z}$ of height $h \leqslant 6$ is isogeneous to the trivial 2-divisible group. This gave the cases $g = 2$ and $g = 3$ of the Shafarevich Conjecture. The method can be explained as follows.

Suppose G is a f.f.g.s. (finite flat group scheme) over $\mathbb{Z}$ such that $2\mathrm{id}_G = 0$. Then

(a) if the order $|G| = 2$ then G is either etale $(\mathbb{Z}/2)_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z} \oplus \mathbb{Z})$ or multiplicative $\mu_2 = \mathrm{Spec}\ \mathbb{Z}[x]/(x^2 - 1)$ f.f.g.s. over $\mathbb{Z}$, [31];

(b) if $|G| = 4$ and $G = \mathrm{Spec}\ A(G)$ is not a product of f.f.g.s. of order 2 then there is a short exact sequence of f.f.g.s.

$$0 \longrightarrow \mu_2 \longrightarrow G \longrightarrow (\mathbb{Z}/2)_{\mathbb{Z}} \longrightarrow 0$$

and $A(G) = A(\mu_2) \oplus \mathbb{Z}[i]$, [1]. In particular, $A(G)_{\mathbb{Q}} \neq \mathbb{Q} \oplus K$, where $[K : \mathbb{Q}] = 3$. (Use that $A(G)_{\mathbb{Q}}$ is etale over $\mathbb{Q}$ and there are no cube field extensions $K/\mathbb{Q}$ unramified outside 2.)

(c) there are similar short exact sequences for f.f.g.s. G over $\mathbb{Z}$ of order 2^n with $n = 3, 4, 5, 6$,

$$0 \longrightarrow \mu_2^a \longrightarrow G \longrightarrow (\mathbb{Z}/2)_{\mathbb{Z}}^b \longrightarrow 0,$$

where $a + b = n$, [2]. This statement is highly non-trivial because the Galois group of the field-of-definition $\mathbb{Q}(G)$ of $\bar{\mathbb{Q}}$-points of f.f.g.s. of order 2^n is not generally soluble if $n \geqslant 4$. On the one hand, we used the Tate formula for the discriminant of $A(G)$ from [31], $v_2(D(A(G)) = d2^n$, where $d = \dim(G \otimes \mathbb{F}_2)$ (it implies that $v_2(D(A(G))) \leqslant 192$ because we can assume that $d \leqslant 3$ by switching, if necessary, from G to its Cartier dual). On the other hand, we used the Odlyzko lower bounds for the minimal discriminants of algebraic number fields, cf. [30, 18, 25];

(d) in the special pre-abelian category of f.f.g.s. G over $\mathbb{Z}$ such that $2\mathrm{id}_G = 0$, one has

$$\mathrm{Ext}(\mu_2, (\mathbb{Z}/2)_{\mathbb{Z}}) = \mathrm{Ext}((\mathbb{Z}/2)_{\mathbb{Z}}, (\mathbb{Z}/2)_{\mathbb{Z}}) = \mathrm{Ext}(\mu_2, \mu_2) = 0.$$

Therefore, the above exact sequences for G and devissage in the pre-abelian category of finite flat 2-group schemes over $\mathbb{Z}$ give the following exact sequence of 2-divisible groups over $\mathbb{Z}$

$$0 \longrightarrow \{\mu_{2^n}\}_{n\geq 1}^a \longrightarrow \mathcal{G} \longrightarrow (\mathbb{Q}_2/\mathbb{Z}_2)^b \longrightarrow 0, \qquad (1.1)$$

where $\mathcal{G}$ is of height $a + b \leqslant 6$ (for more details about devissage in pre-abelian categories cf. Appendix, especially Theorem A.1);

(e) such 2-divisible group $\mathcal{G}$ never comes from a non-trivial abelian scheme A over $\mathbb{Z}$. Otherwise, looking at dimensions we obtain $b \neq 0$, but the exact sequence of 2-divisible groups from (d) splits over $\mathbb{F}_2$ and, therefore, A has infinitely many $\mathbb{F}_2$-points. The contradiction.

The above method does not work in higher dimensions.

Indeed, suppose A is an abelian scheme over $\mathbb{Z}$ and $G = \mathrm{Ker}(2\mathrm{id}_A)$ is a group scheme of points of order $\leqslant 2$ on A. Then $|G| = 2^{2g}$, $\dim(G \otimes \mathbb{F}_2) = g$ and Tate's formula gives $v_2(D(A(G))^{1/2g}) = g$. Note that $A(G) \otimes \mathbb{Q}$ is the product of algebraic number fields (because $G \otimes \mathbb{Q}$ is etale) and these fields are unramified outside 2 (because $G \otimes \mathbb{Z}_l$ is etale if $l \neq 2$). Therefore, the normalized discriminant of $A(G)$ equals 2^g and tends to infinity if $g \to \infty$.

On the other hand, if $\mathbb{Q}(G)$ is the field-of-definition of $\bar{\mathbb{Q}}$-points of G, then $\mathrm{Gal}(\mathbb{Q}(G)/\mathbb{Q}) \subset \mathrm{SL}(2g, \mathbb{F}_2)$ is not generally soluble if $g \geqslant 2$, and the only global idea we can use in this situation is related to lower bounds of minimal discriminants of algebraic number fields. The best known bounds are the Odlyzko estimates and they are given by the tables of real numbers $\{d_N \mid N \in \mathbb{N}\}$ such that if $[K : \mathbb{Q}] = N$ then $|D(K/\mathbb{Q})|^{1/N} \geq d_N$. For large N, $d_N \approx d_\infty \approx 22.3$; under *GRH* there are better estimates $\{d_N^* \mid N \in \mathbb{N}\}$ in this case $d_\infty^* \approx 44.76$, [30, 18, 25].

Unfortunately, an analogue of Odlyzko estimates under additional assumption that $K/\mathbb{Q}$ is ramified only over 2, does not exist. Nonetheless, $A(G)$ is considerably smaller than its integral closure and Tate's formula can be replaced by a much better upper estimate for the 2-adic valuation of the normalized discriminant of $\mathbb{Q}(G)$. The evidence for its existence is illustrated in the next section.

1.2. The Shafarevich Conjecture, the ordinary case

Suppose our abelian variety A has good ordinary reduction at 2. Then:

(a) $G := \mathrm{Ker}(2\mathrm{id}_A)$ is a f.f.g.s. over $\mathbb{Z}$ of order 2^{2g};

(b) there is a short exact sequence of f.f.g.s. over $\mathbb{Z}_2$

$$0 \longrightarrow H^{mult} \longrightarrow G \otimes_{\mathbb{Z}} \mathbb{Z}_2 \longrightarrow H^{et} \longrightarrow 0,$$

where H^{mult} is multiplicative and H^{et} is etale group schemes over $\mathbb{Z}_2$ of order 2^g;

(c) because $H^{et} \otimes W(\bar{\mathbb{F}}_2) = \prod_j (\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)}$ and $H^{mult} \otimes W(\bar{\mathbb{F}}_2) = \prod_i \mu_{2,W(\bar{F}_2)}$, we have

$$G \otimes W(\bar{\mathbb{F}}_2) = \sum_{i,j} G_{ij} \in \oplus_{i,j}\mathrm{Ext}((\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)}, \mu_{2,W(\bar{\mathbb{F}}_2)}),$$

where for all i, j, there are short exact sequences of f.f.g.s.

$$0 \longrightarrow \mu_{2,W(\bar{\mathbb{F}}_2)} \longrightarrow G_{ij} \longrightarrow (\mathbb{Z}/2)_{W(\bar{\mathbb{F}}_2)} \longrightarrow 0;$$

(d) the field-of-definition of geometric points of G_{ij} over the maximal unramified extension $\mathbb{Q}_{2,ur}$ of $\mathbb{Q}_2$, is $\mathbb{Q}_{2,ur}(\sqrt{v_{ij}})$, where all v_{ij} are principal units in $\mathbb{Q}_{2,ur}$, cf. Appendix by J.Tate in [26]. Therefore, for all $v > 1$, the higher ramification subgroups $\Gamma_{\mathbb{Q}_2}^{(v)}$ of $\Gamma_{\mathbb{Q}_2} = \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2)$ act trivially on the field-of-definition $\mathbb{Q}_2(G)$ of all $\bar{\mathbb{Q}}_2$-points of G;

(e) the triviality of $\Gamma_{\mathbb{Q}_2}^{(v)}$-action, where $v > 1$, implies the inequality $|D(\mathbb{Q}(G)/\mathbb{Q})|^{1/[\mathbb{Q}(G):\mathbb{Q}]} < 2^2$ (e.g. use Prop 9.4 of Ch. 1, [12]). But the Odlyzko estimate $d_4 < 4$ and we obtain $[\mathbb{Q}(G) : \mathbb{Q}] < 4$. Therefore, $\mathbb{Q}(G) \subset \mathbb{Q}(i)$, we can use devissage to obtain the exact sequence (1.1) for $a = b = g$ and finish the proof similarly to the case of small g.

In the above discussion, the prime number 2 can be replaced by arbitrary prime number p. If $A \otimes \mathbb{F}_p$ is ordinary and $G = \mathrm{Ker}(p\,\mathrm{id}_A)$ then for $v > 1/(p-1)$, the ramification subgroups $\Gamma_{\mathbb{Q}_p}^{(v)}$ act trivially on $\mathbb{Q}_p(G)$ and using the Odlyzko estimates we can see that for $3 \leqslant p \leqslant 17$, $\mathbb{Q}(G) \subset \mathbb{Q}(\sqrt[p]{1})$. This implies that G is the product of constant etale and multiplicative f.f.g.s. over $\mathbb{Z}$, the corresponding p-divisible group of A will be just the product of several copies of trivial etale $(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbb{Z}}$ and multiplicative $\{\mu_{2^n,\mathbb{Z}}\}_{n\geqslant 1}$ p-divisible groups over $\mathbb{Z}$ and, therefore, such abelian variety does not exist.

The above case of the Shafarevich Conjecture was not published but gave a right direction towards the proof of the general case.

1.3. The Shafarevich Conjecture, the general case

In this case the same ramification estimates are proved in the general situation [21, 5]: if G is a finite flat group scheme over $W(k)$, where k is a perfect field of characteristic p, $p\,\mathrm{id}_G = 0$ and $\mathrm{Frac}\, W(k) = K$ then the higher ramification subgroups $\Gamma_K^{(v)}$ act trivially on the field-of-definition of $\bar{K}$-points of G for all $v > 1/(p-1)$.

Essentially, Fontaine found ramification estimates for any finite flat p-group schemes over the valuation ring O_L of complete discrete valuation field $L \supset \mathbb{Q}_p$. His method uses the rigidity properties of p-divisible groups defined over valuation rings and a very elegant interpretation of Krasner's Lemma. The methods in [3, 5] are much more computational and use Fontaine's theory of f.f.g.s. over Witt vectors, [20]. In Section 3 we present an alternative proof of ramification estimates. It works also equally well for the subquotients of crystalline and semi-stable p-adic representations.

In our approach from [3, 5] we treated systematically also the case $p = 2$. Here the category of f.f.g.s. over $W(k)$ is not abelian contrary to the case $p \neq 2$, but one can still proceed with the devissage. This gave us not only the bigger list of algebraic number fields where the Shafarevich conjecture about the non-existence of abelian varieties with everywhere good reduction holds. Our main idea [4] of removing the restriction to unipotent objects in Fontaine's classification of 2-group schemes in [20] gave later a right approach to the constructions of modifications of the Fontaine–Laffaille [6] and Breuil [11] functors. These modifications allow us to obtain the ramification estimates for all modulo p subquotients of representations with HT weights from $[0, p)$. They also provide us with the nullity of some groups of extensions in the category of Galois modules appearing as such subquotients. As a matter of fact, these two key ingredients resulted finally in proving Theorem 1.1 and part (b) of Theorem 1.2.

2. The functor $\mathcal{CV}^*$

Let $\mathcal{W}_1 = k[[u]]$, where u is an indeterminate. Denote by σ the automorphism of k induced by the p-th power map on k and agree to use the same symbol for the continuous extension of σ to $\mathcal{W}_1$ such that $\sigma(u) = u^p$. Denote by $N : \mathcal{W}_1 \longrightarrow \mathcal{W}_1$ the unique continuous k-differentiation such that $N(u) = -u$.

2.1. Categories of filtered modules

Introduce the following categories:

- the category $\widetilde{\underline{\mathcal{L}}}_0^*$ – its objects are $\mathcal{L} = (L, F(L), \varphi)$, where L and $F(L)$ are $\mathcal{W}_1$-modules, $L \supset F(L)$ and $\varphi : F(L) \longrightarrow L$ is a σ-linear morphism of $\mathcal{W}_1$-modules; the morphisms are $\mathcal{W}_1$-linear maps of filtered modules which commute with the corresponding σ-linear maps φ;
- the category $\widetilde{\underline{\mathcal{L}}}^*$ – its objects are $\mathcal{L} = (L, F(L), \varphi, N)$, where $(L, F(L), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$ and $N : L \longrightarrow L/u^pL$ is such that for $w \in \mathcal{W}_1$ and $l \in L$, $N(wl) = N(w)l + wN(l)$ (we use the same notation l for the image of l in L/u^pL); the morphisms are the morphisms from $\widetilde{\underline{\mathcal{L}}}_0^*$ which commute with the corresponding differentiations N;
- the category $\underline{\mathcal{L}}_0^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}_0^*$ consisting of $\mathcal{L} = (L, F(L), \varphi)$ such that the module L is free of finite rank, $u^{p-1}L \subset F(L)$ and the natural embedding $\varphi(F(L)) \subset L$ induces the identification
$$\varphi(F(L)) \otimes_{\sigma(\mathcal{W}_1)} \mathcal{W}_1 = L;$$
- the category $\underline{\mathcal{L}}^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}^*$ consisting of $\mathcal{L} = (L, F(L), \varphi, N)$ such that $(L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$, for any $l \in F(L)$, one has $uN(l) \in F(L) \bmod u^pL$ and $N(\varphi(l)) = \varphi(uN(l))$ (we use the same notation φ for the morphism $\varphi \bmod u^pL$).

The above categories are analogs of the categories of filtered modules from [14], Subsection 2.1.2, but we work with the category of $\mathcal{W}_1$-modules. (Breuil uses modules over the appropriate divided power envelope of $W(k)[[u]]$.) Note that in the context of $\mathcal{W}_1$-modules the monodromy operator N can't be defined as a map with values in L. In [11], Subsection 1.1, we proved that N can be defined as a map from L to $L/u^{2p}L$ and it appears as a unique lift of its reduction $N_1 = N \bmod u^p L$. (We used the existence of such a lift when proving in [11] that the category $\underline{\mathcal{L}}^*$ is pre-abelian; we also need this property when defining the functor $\mathcal{V}^*$ in Subsection 2.3 below.) In this chapter we use the notation N for this $(\bmod\, u^p)$-map N_1;

- the category $\underline{\mathcal{L}}^*_{cr}$ is a full subcategory in $\underline{\mathcal{L}}^*$ consisting of the objects $(L, F(L), \varphi, N)$ such that $N(\varphi(F(L))) = 0$.

For obvious reasons, $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*_{cr}$ is completely determined by $(L, F(L), \varphi) \in \underline{\mathcal{L}}^*_0$. Note that the category $\underline{\mathcal{L}}^*_{cr}$ is very closely related to the category of Fontaine–Laffaille modules, cf. [11], Subsection 1.3.

According to above definitions the objects of the categories $\underline{\mathcal{L}}^*_0$, $\underline{\mathcal{L}}^*$ and $\underline{\mathcal{L}}^*_{cr}$ are filtered free $\mathcal{W}_1$-modules with additional structures. The category of filtered free $\mathcal{W}_1$-modules is a typical example of a special pre-abelian category, i.e. it is an additive category with kernels and cokernels and nicely behaving bifunctor Ext, cf. Appendix. In Subsection 1.1 of [11] we verified that $\underline{\mathcal{L}}^*_0$, $\underline{\mathcal{L}}^*$ and $\underline{\mathcal{L}}^*_{cr}$ inherit the property of being special pre-abelian.

There are the concepts of etale, unipotent, connected and multiplicative objects in our categories defined in the following way; for more details cf. Subsection 1.2 of [11].

Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$.

Introduce a σ-linear map $\phi : L \longrightarrow L$ via $\phi : l \mapsto \varphi(u^{p-1}l)$. The module $\mathcal{L}$ is etale (resp., connected) if $\phi \bmod u$ is invertible (resp., nilpotent) on L/uL. Denote by $\underline{\mathcal{L}}^{*et}$ (resp, $\underline{\mathcal{L}}^{*c}$) the full subcategory of $\underline{\mathcal{L}}^*$ consisting of etale (resp. connected) objects. Then any $\mathcal{L} \in \underline{\mathcal{L}}^*$ contains a unique maximal etale subobject $(\mathcal{L}^{et}, i^{et})$ and a unique maximal connected quotient object $(\mathcal{L}^c, j^c)$ and the sequence

$$0 \longrightarrow \mathcal{L}^{et} \xrightarrow{i^{et}} \mathcal{L} \xrightarrow{j^c} \mathcal{L}^c \longrightarrow 0$$

is short exact.

Note that $\varphi(F(L))$ is a $\sigma(\mathcal{W}_1)$-module and $L = \varphi(F(L)) \otimes_{\sigma(\mathcal{W}_1)} \mathcal{W}_1$. If $l \in L$ and for $0 \leqslant i < p$, the elements $l^{(i)} \in F(L)$ are such that $l = \sum_{0 \leqslant i < p} \varphi(l^{(i)}) \otimes u^i$, set $V(l) = l^{(0)}$. Then $V \bmod u$ is a σ^{-1}-linear endomorphism of the k-vector space L/uL.

The module $\mathcal{L}$ is multiplicative (resp., unipotent) if V mod u is invertible (resp., nilpotent) on L/uL. Denote by $\underline{\mathcal{L}}^{*m}$ (resp, $\underline{\mathcal{L}}^{*u}$) the full subcategory of $\underline{\mathcal{L}}^{*}$ consisting of multiplicative (resp. unipotent) objects. Then any $\mathcal{L} \in \underline{\mathcal{L}}^{*}$ contains a unique maximal multiplicative quotient object $(\mathcal{L}^m, j^m)$ and a unique maximal unipotent subobject $(\mathcal{L}^u, i^u)$ and the sequence

$$0 \longrightarrow \mathcal{L}^u \xrightarrow{i^u} \mathcal{L} \xrightarrow{j^m} \mathcal{L}^m \longrightarrow 0$$

is short exact.

Note that $\underline{\mathcal{L}}^{*c}$ and $\underline{\mathcal{L}}^{*u}$ are abelian categories: it follows easily from the description of simple objects of $\underline{\mathcal{L}}^{*}$ in Subsection 1.4 of [11].

2.2. The object $\mathcal{R}^0_{st} \in \widetilde{\underline{\mathcal{L}}}^*$

Let $R = \varprojlim_n (\bar{O}/p)_n$ be Fontaine's ring; it has a natural structure of k-algebra via the map $k \longrightarrow R$ given by $\alpha \mapsto \varprojlim_n([\sigma^{-n}\alpha] \mathrm{mod}\, p)$, where $[\gamma] \in W(k) \subset \bar{O}$ denotes the Teichmüller representative of $\gamma \in k$. Let m_R be the maximal ideal of R.

Choose $x_0 = (x_0^{(n)} \bmod p)_{n\geqslant 0} \in R$ and $\varepsilon = (\varepsilon^{(n)} \mathrm{mod}\, p)_{n\geqslant 0}$ such that for all $n \geqslant 0$, $x_0^{(n+1)p} = x_0^{(n)}$ and $\varepsilon^{(n+1)p} = \varepsilon^{(n)}$ with $x_0^{(0)} = -p$, $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$. Denote by v_R the valuation on R such that $v_R(x_0) = 1$.

Let Y be an indeterminate.

Consider the divided power envelope $R\langle Y\rangle$ of $R[Y]$ with respect to the ideal (Y). If for $j \geqslant 0$, $\gamma_j(Y)$ is the j-th divided power of Y then $R\langle Y\rangle = \oplus_{j\geqslant 0} R\gamma_j(Y)$. Denote by R_{st} the completion $\prod_{j\geqslant 0} R\gamma_j(Y)$ of $R\langle Y\rangle$ and set, $\mathrm{Fil}^p R_{st} = \prod_{j\geqslant p} R\gamma_j(Y)$. Define the σ-linear morphism of the R-algebra R_{st} by the correspondence $Y \mapsto x_0^p Y$; it will be denoted below by the same symbol σ.

Introduce a $\mathcal{W}_1$-module structure on R_{st} by the k-algebra morphism $\mathcal{W}_1 \longrightarrow R_{st}$ such that $u \mapsto \iota(u) := x_0 \exp(-Y) = x_0 \sum_{j\geqslant 0}(-1)^j \gamma_j(Y)$.

Set $F(R_{st}) = \sum_{0\leqslant i<p} x_0^{p-1-i} R\gamma_i(Y) + \mathrm{Fil}^p R_{st}$.

Define the continuous σ-linear morphism of R-modules $\varphi : F(R_{st}) \longrightarrow R_{st}$ by setting for $0 \leqslant i < p$, $\varphi(x_0^{p-1-i}\gamma_i(Y)) = \gamma_i(Y)(1-(i/2)x_0^p Y)$, and for $i \geqslant p$, $\varphi(\gamma_i(Y)) = 0$.

Let N be a unique R-differentiation of R_{st} such that $N(Y) = 1$.

Note that $(R_{st}, F(R_{st}), \varphi, N)$ is not an object of $\widetilde{\underline{\mathcal{L}}}^*$, e.g. φ is not a σ-linear morphism of $\mathcal{W}_1$-modules. Nevertheless, all appropriate compatibilities between above introduced additional structures on R_{st} hold modulo $x_0^{2p} R_{st}$, cf. Proposition 2.1 in [11], and we can introduce

$$\mathcal{R}^0_{st} = (R^0_{st}, F(R^0_{st}), \varphi, N) \in \widetilde{\underline{\mathcal{L}}}^*,$$

where $R^0_{st} = R_{st} \bmod x_0^p \mathrm{m}_R$ and $F(R^0_{st}) = F(R_{st}) \bmod x_0^p \mathrm{m}_R$ with the appropriate induced maps φ and N.

In our theory $\mathcal{R}^0_{st}$ plays a role of the ring $\hat{A}_{st}$ from the theory of p-adic semi-stable representations [14], Subsection 3.1.1. In particular, R^0_{st} can be provided with continuous Galois action as follows. For any $\tau \in \Gamma_K$, let $k(\tau) \in \mathbb{Z}$ be such that $\tau(x_0) = \varepsilon^{k(\tau)} x_0$ and let $\widetilde{\log}(1+X) = X - X^2/2 + \cdots - X^{p-1}/(p-1)$ be the truncated logarithm. Define a map $\tau : R_{st} \longrightarrow R_{st}$ by extending the natural action of τ on R and setting for all $j \geqslant 0$,

$$\tau(\gamma_j(Y)) := \sum_{0 \leqslant i \leqslant \min\{j, p-1\}} \gamma_{j-i}(Y)\gamma_i(\widetilde{\log}\varepsilon).$$

Then the correspondences $\gamma_j(Y) \mapsto \tau(\gamma_j(Y))$ induce a Γ_K-action on the $\mathcal{W}_1$-algebra R^0_{st} which extends the natural Γ_K-action on R and respects the structure of $\mathcal{R}^0_{st}$ as an object of the category $\widetilde{\underline{\mathcal{L}}}^*$, cf. Proposition 2.2 in [11].

2.3. The functor $\mathcal{V}^*$

For any $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$, consider the Γ_K-module $\mathcal{V}^*(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}, \mathcal{R}^0_{st})$. Note that in this definition we need N to be defined slightly better than just modulo $u^p L$ (we work modulo $x_0^p \mathrm{m}_R$ rather than modulo $x_0^p R$) but such lift exists and unique, cf. Subsection 2.1. The Galois module $\mathcal{V}^*(\mathcal{L})$ can be studied via the following method from [15], Subsection 2.3.

Let $\mathcal{R}^0 = (R^0, F(R^0), \varphi) \in \widetilde{\underline{\mathcal{L}}}^*_0$, where $R^0 = R/x_0^p \mathrm{m}_R$, $F(R^0) = x_0^{p-1} R^0$, the $\mathcal{W}_1$-module structure on R^0 is given via $u \mapsto x_0$ and ϕ is induced by the map $r \mapsto r^p / x_0^{p(p-1)}$, $r \in x_0^{p-1} R$.

If $f \in \mathcal{V}^*(\mathcal{L})$ and $i \geqslant 0$, introduce k-linear morphisms $f_i : L \longrightarrow R^0$ such that for any $l \in L$, $f(l) = \sum_{i \geqslant 0} f_i(l)\gamma_i(Y)$. The correspondence $f \mapsto f_0$ gives the homomorphism of abelian groups $\mathrm{pr}_0 : \mathcal{V}^*(\mathcal{L}) \longrightarrow \mathcal{V}^*_0(\mathcal{L}) := \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*_0}(\mathcal{L}, \mathcal{R}^0)$. Then, cf. Subsection 2.2 of [11],

- pr_0 is an isomorphism of abelian groups;
- if $\mathrm{rk}_{\mathcal{W}_1} L = s$ then $|\mathcal{V}^*_0(\mathcal{L})| = p^s$.

Therefore, $\mathcal{V}^*$ is an exact functor from $\underline{\mathcal{L}}^*$ to the category of finite $\mathbb{F}_p[\Gamma_K]$-modules.

Introduce the ideal $\widetilde{J} = \sum_{0 \leqslant i < p} x_0^{p-i} \mathrm{m}_R \gamma_i(Y) + \mathrm{Fil}^p R^0_{st}$ in R^0_{st}. Then $F(R^0_{st}) \supset \widetilde{J}$ and $\varphi|_{\widetilde{J}}$ is nilpotent. Therefore, we can introduce $\widetilde{\mathcal{R}}^0_{st} = (R^0_{st}/\widetilde{J}, F(R^0_{st})/\widetilde{J}, \varphi \bmod \widetilde{J}) \in \widetilde{\underline{\mathcal{L}}}^*_0$, there is a natural projection $\mathcal{R}^0_{st} \longrightarrow \widetilde{\mathcal{R}}^0_{st}$ in $\widetilde{\underline{\mathcal{L}}}^*_0$ and for any $\mathcal{L} \in \underline{\mathcal{L}}^*_0$, $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*_0}(\mathcal{L}, \mathcal{R}^0_{st}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*_0}(\mathcal{L}, \widetilde{\mathcal{R}}^0_{st})$. This implies the following description of the Γ_K-modules $\mathcal{V}^*(\mathcal{L})$, $\mathcal{L} \in \underline{\mathcal{L}}^*$,