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Caroline Meyrick and Kwame Dwamena
Excerpt
More information

1 Number and algebra

What have you done today? Did you:

- solve a puzzle?
- look at something on the internet?
- take some medication?
- use your cell phone?

What might you be doing in ten years' time? Do you think that you will be:

- working in a design office?
- using spreadsheets to analyse large amounts of data?
- involved in making financial or investment decisions?
- looking into the global logistics of moving goods and materials?

In all of the above activities, algorithms are used to calculate the most efficient way of solving a problem. An **algorithm** can be defined as *a detailed set of instructions that leads to a predictable result or a set of instructions that helps you to finish a task as efficiently as possible.*

The mathematical skills of manipulating numbers and using algebra provide a powerful tool in devising, understanding and applying algorithms.

Prior learning topics

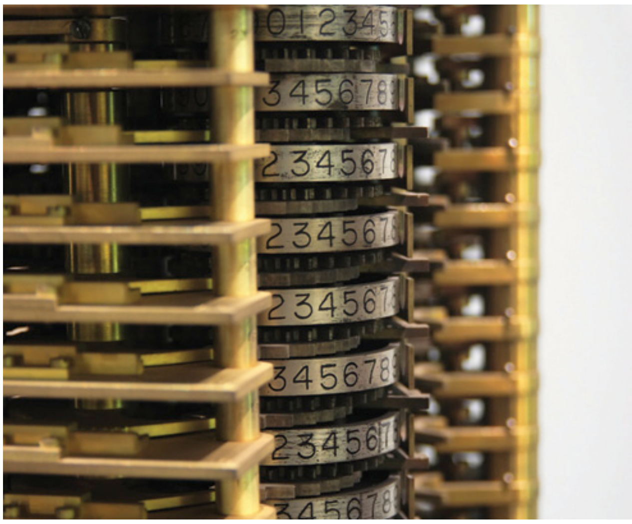
It will be easier to study this topic if you:

- are comfortable with the four operations of arithmetic (addition, subtraction, multiplication, division)
- understand the **BIDMAS** order of operations (**B**rackets, **I**ndices, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction)
- can use integers, decimals and fractions in calculations
- can recognise prime numbers, factors and multiples
- know some simple applications of ratio, percentage and proportion
- can identify intervals on the real number line
- are able to evaluate simple algebraic expressions by substitution
- are comfortable with basic manipulations of algebraic expressions such as factorisation, expansion and rearranging formulae
- understand how to use the inequalities $<, \leq, >, \geq$
- are familiar with commonly accepted world currencies such as the euro, United States dollar and Japanese yen.

Chapter 1 Number

In this chapter you will learn:

- about the different types of numbers
- how to make approximations
- about estimating and how to check solutions using your GDC
- how to express very small and very large numbers and perform operations with them
- about SI (Système International) units.



An Analytical Engine

Ada Lovelace had a very clear idea of the role of the analytical engine, saying: 'The Analytical Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform. It can follow analysis; but it has no power of anticipating any analytical revelations or truths. Its province is to assist us in making available what we are already acquainted with.' This is 'Note G' from Ada Lovelace's translation and commentary of a 1842 document called *Sketch of the Analytical Engine* by L.F. Menabrea (information sourced from www.allonrobots.com/ada-lovelace.html). This is a good description of all computers and calculating machines.

Charles Babbage (1791–1871) is described in some histories as 'the father of computing'. A mathematician and inventor, he was looking for a method to improve the accuracy of mathematical tables. These lists of numbers included squares and square roots, logarithms and trigonometric ratios, and were used by engineers, navigators and anyone who needed to perform complex arithmetical calculations. The tables were notorious for their inaccuracy, so Babbage designed a machine to re-calculate the numbers mechanically. In 1822, the Royal Society approved his design, and the first 'difference engine' was built at the inventor's home in London.

Babbage went on to develop an improved machine, called an 'Analytical Engine', which is now seen as the first step towards modern computers. He worked on this machine with Ada Lovelace (1815–1852), who invented the punched cards that were used to 'programme' the machine. Lovelace is considered to be the first computer programmer.

1.1 Different types of numbers

The natural numbers, ℕ

The **natural numbers** (ℕ) are the counting numbers, the first numbers that people learn and use.

1, 2, 3, 4, 5, … are all counting numbers.

–1, –2, –3, … are negative numbers and therefore are **not** natural numbers.

1.5, 2.3, 6.7 are not whole numbers and therefore are **not** natural numbers.

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When small children learn to count, they soon realise that '3' will always mean the same quantity, three, whether they are counting apples, sheep or chairs.

Natural numbers are also the numbers typically used for comparison. For example, you might say, 'I have read all seven Harry Potter books; the second and the seventh were my favourites.'

The place of zero among the natural numbers is a point of debate. Not all definitions of the natural numbers include zero. In this IB course, the natural numbers do include the number zero.

Natural numbers can be shown on a **number line**:

We can also write the natural numbers as follows:

$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$, where $\mathbb{N}$ is the symbol for natural numbers.

The curly brackets { } enclose all the numbers represented by the symbol $\mathbb{N}$. These brackets signify that the natural numbers form a **set** (or collection).

FF *Sets and set notation are explained in Chapter 8.*

The integers, $\mathbb{Z}$

Natural numbers can only be used for counting in one direction: left to right along the number line, starting from zero. It is often useful to be able to count down to below zero. This is where a larger set of numbers, called the 'integers', comes in.

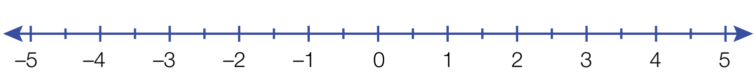
The **integers** are defined as all whole numbers: positive, negative and zero.

−79, −2 and 1 0001 are all integers.

−9.99, $1\frac{1}{4}$ and 1 0001.4 have decimal or fractional parts and are therefore **not** integers.

The number line is infinitely long and extends from negative infinity to positive infinity. The concept of **infinity** is very important to mathematicians, but it is difficult to define. Imagine a line, divide it in half, then divide in half again, and continue in this way. No matter how many times you divide the line in half, you can always divide it again. So you can do an 'infinite' number of divisions. The symbol for infinity is ∞.

Integers can also be shown on a number line:



We can use set notation to write the integers, as we did for natural numbers:

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$, where $\mathbb{Z}$ is the symbol for integers.

$\mathbb{Z}^{+}$ is defined as the set of all positive whole numbers, or integers greater than zero.

$\mathbb{Z}^{-}$ is defined as the set of all negative whole numbers, or integers less than zero.

The symbol $\mathbb{Z}$ comes from 'Zahlen', the German word for numbers.

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The rational numbers, $\mathbb{Q}$

A **rational number** is a number that results when one integer is divided by another. Dividing one integer by another creates a **ratio**, which is where the term '**ratio**nal numbers' comes from.

If q is a rational number, then $q = \frac{a}{b}$, where a and b are both integers, with $b \neq 0$ (a is called the **numerator** and b is called the **denominator**).

$-\frac{7}{2}, \frac{1}{5}, \frac{28}{9}, 5\frac{3}{4}$ are all rational numbers.

hint

$\mathbb{Q}$ stands for **quotient**. A quotient is the result of a division.

Note that mixed numbers such as $5\frac{3}{4}$ are also rational numbers. This is true because you can rewrite them as **improper fractions**, e.g. $5\frac{3}{4} = \frac{23}{4}$.

Rational numbers are often written as decimals, which can make it less obvious that they are rational numbers. For example:

$-3.5, 0.2, 3.111111111\ldots$ are all rational numbers.

hint

Remember that a mixed number is made up of a whole number and a fraction, e.g. $1\frac{2}{3}$. An improper fraction is a fraction where the numerator is larger than the denominator, e.g. $\frac{5}{4}$.

You can check by writing each as a fraction:

$-3.5 = -\frac{7}{2}$, so -3.5 is a rational number.

$0.2 = \frac{2}{10} = \frac{1}{5}$, so 0.2 is a rational number.

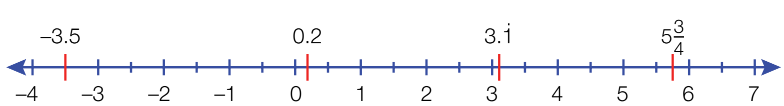
$3.111111111\ldots = \frac{28}{9}$, so $3.111111111\ldots$ is a rational number.

Numbers like $3.111111111\ldots$, where a digit (or group of digits) repeats forever, are called 'recurring decimals' and are often written with a dot above the number that repeats, e.g. $3.\dot{1}$. All recurring decimals are rational numbers.

hint

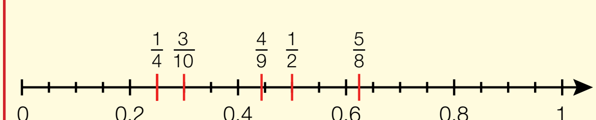
All integers are rational numbers too, e.g. $5 = \frac{5}{1}$ is a fraction with denominator 1.

Rational numbers can also be shown on a number line:



Worked example 1.1

Q. Find a rational number between $\frac{1}{4}$ and $\frac{5}{8}$.



There are an infinite number of fractions to choose from. A number line can help you make a correct choice.

A. Possible fractions include $\frac{3}{10}, \frac{4}{9}, \frac{1}{2}$

The irrational numbers

An **irrational number** is a number that **cannot** be written as a fraction. The decimal part of an irrational number has no limit to the number of digits it contains and does not show a repeating pattern.

A well-known example of an irrational number is π (pi). You know this as the ratio of the circumference of a circle to its diameter: $\pi = 3.14159265\ldots$

Modern computers enable mathematicians to calculate the value of π to many millions of digits, and no repeats of groups of digits have been found!

Other irrational numbers include ϕ (the 'golden ratio') and the square roots of **prime numbers**:

$$\phi = 1.6180339887498948482 0\ldots$$

$$\sqrt{2} = 1.414213562373\ldots$$

$$\sqrt{7} = 2.6457513110645\ldots$$

Irrational numbers do not have their own symbol; however, they are sometimes represented by $\bar{\mathbb{Q}}$. The bar above the $\mathbb{Q}$ indicates that the irrational numbers form the **complement** (opposite) of $\mathbb{Q}$ (the rational numbers). Why do you think irrational numbers have not been given their own symbol?

The real numbers, $\mathbb{R}$

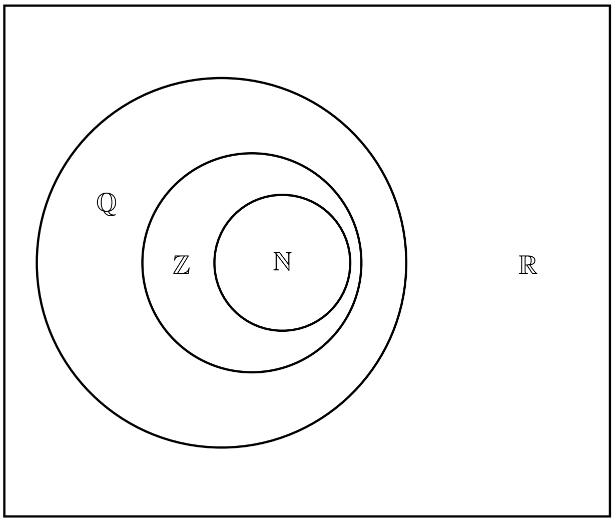
The **real numbers** are all the numbers that can be represented on a number line.

They include the natural numbers, integers, rational numbers and irrational numbers.

It may be easier to think of the various types of real numbers in the form of a **Venn diagram**.

A Venn diagram uses shapes (usually circles) to illustrate mathematical ideas. Circles may overlap or lie inside each other. In the example below, $\mathbb{N}$ is inside $\mathbb{Z}$ because all natural numbers are integers.

FF *You will learn about Venn diagrams in Chapter 8.*



Irrational numbers are included in the definition of real numbers. In the Venn diagram they are represented by the region **outside** the natural numbers, integers and rational numbers.

It can be proved that there are an infinite number of real numbers. Georg Cantor constructed the proof in 1874. It is a powerful proof, surprisingly simple, and worth reading about and understanding. The ideas behind the proof are very profound and have had an important influence on the work of mathematicians following Cantor.

exam tip

In this course, all the numbers you will encounter are real numbers.

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The history of numbers follows the development of human history. The modern system of numbers is based on Arabic notation, and also uses ideas from Indian mathematics, such as zero.

Worked example 1.2

Q. Mark each cell to indicate which number set(s) the number belongs to.

	-2	$\frac{3}{7}$	$\sqrt{13}$	3π	10 000
Irrational					
$\mathbb{N}$					
$\mathbb{Z}$					
$\mathbb{Q}$					
$\mathbb{R}$					

A.

	-2	$\frac{3}{7}$	$\sqrt{13}$	3π	10 000
Irrational			×	×	
$\mathbb{N}$					×
$\mathbb{Z}$	×				×
$\mathbb{Q}$	×	×			×
$\mathbb{R}$	×	×	×	×	×

The cells have been filled in to show that:

- -2 is an integer, a rational number and a real number
- $\frac{3}{7}$ is a rational number and a real number
- $\sqrt{13}$ is an irrational number and a real number
- 3π is an irrational number and a real number
- 10 000 is a natural number, an integer, a rational number and a real number.

Do you think that numbers have always been in existence, waiting to be discovered, or are they an invention of mathematicians?

Worked example 1.3

Q. Look at the list of numbers:

$\sqrt{5}, -\frac{3}{7}, \pi, -5, 7, 2^3.$

(a) Which numbers are integers?

(b) Which numbers are both rational and negative?

(c) Which numbers are not rational?

(d) Which numbers are not natural?

A. (a) $-5, 7, 2^3$

These are all integers; note that $2^3 = 8$.

(b) $-\frac{3}{7}$ and -5

Remember that -5 is also a rational number $\left(-\frac{5}{1}\right)$.

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$\sqrt{5}$ cannot be written as a fraction, and as a decimal it does not have a limit or a recurring pattern; the same is true for π.

These are not counting numbers, so they are not natural.

continued . . .

(c) $\sqrt{5}$ and π

(d) $\sqrt{5}, -\frac{3}{7}, \pi, -5$

Exercise 1.1

1. Look at each of the following statements and decide whether it is true or false. If it is false, give the correct statement.

(a) 2.4 is a rational number.

(b) $6 + (-2)$ gives an answer that is a natural number.

(c) $\sqrt{17}$ is a rational number.

(d) 1.51 is an irrational number.

(e) 5π is a real number.

(f) An irrational number is never a real number.

(g) If you add two integers, the answer will not be an integer.

(h) If you divide one integer by another, the answer is a real number.

Srinivasa Ramanujan was a mathematical genius. Born in 1887 near Madras in India, he was fascinated by numbers and made many significant contributions to the theory of numbers. He is also well known for his collaboration with the famous mathematician G. H. Hardy of the University of Cambridge.

2. Write down a number that is:

(a) a real number and an integer

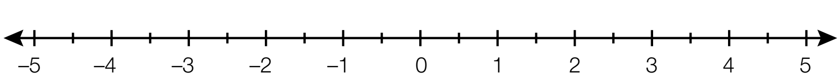
(b) a rational number, but not an integer

(c) a real number and an irrational number

(d) a natural number that is also rational.

3. Copy the number line below, and put these numbers in the correct place on the number line:

$$\frac{12}{13}, -\sqrt{3}, 3.1, -1+\pi, -4.2, -4.25, \sqrt{13}$$



4. (a) Put the following numbers in ascending order:

$$12, -5.\dot{1}, \sqrt{3}, \sqrt{2}, -2, 0, \frac{6}{7}, -2.5$$

(b) Write down the natural numbers in the list.

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(c) Write down the integers in the list.

(d) Write down the rational numbers in the list.

(e) Which numbers have you not written down? Why?

5. Look at each of the following statements, and use the given words to fill in the blanks, making a correct sentence.

rational, irrational, real, negative, natural, integer

(a) If you add two natural numbers, the answer is a(n) .... number.

(b) If you add two rational numbers, the answer can be a(n) .... number, a(n) .... (number) or a(n) .... number

(c) If you add a negative integer to another negative integer, the answer is a(n) ....

(d) If you add a natural number to an irrational number, the answer is a(n) .... number.

(e) If you add a natural number that is greater than 12 to an integer between −10 and zero, the answer is a(n) ....

6. (a) Find a rational number between $\frac{4}{5}$ and $1\frac{3}{4}$.

(b) Find a rational number between $2\frac{2}{3}$ and $3\frac{3}{4}$.

7. Look at the list of numbers below, and use it to answer the following questions:

43, 2, 21, 15, –6, 17, 6, –4, 13

(a) Write down the prime numbers.

(b) Write down the multiples of three.

(c) Write down the even numbers.

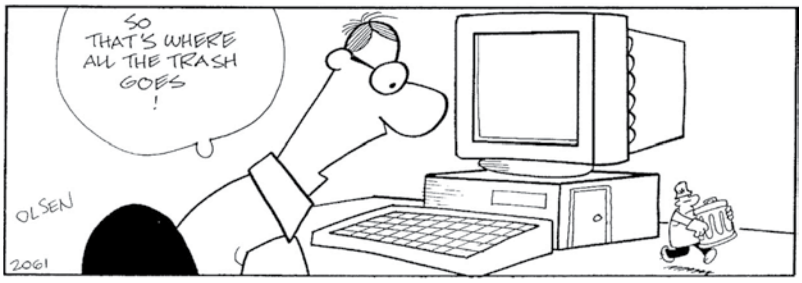
(d) Which number have you written for both (a) and (c)?

1.2 Approximation and estimation

Modern calculators and computers allow us to calculate with great precision. However, in everyday life most people use estimates and few people are comfortable with very large, very small, or very long numbers.

With the increasing use of calculators and computers, it has become more important to know how to estimate a rough answer and to 'round' the results obtained by technology so that we can use them sensibly.

There is a well-known saying about modern technologies: 'Garbage **in**, garbage **out**'. This means that you need to put sensible input into your calculator or computer, in order to be able to make sense of what you get as an output.



We also need to agree upon our methods of **rounding**; for example, if one shopkeeper always rounds her prices **down** to the nearest cent, while her rival always rounds his prices **up** to the nearest cent, what will be the result?

Your graphical display calculator (GDC) is a piece of modern technology that is fundamental to this course. You cannot achieve a good understanding of the material without it! It is important that you are able to:

- estimate a rough answer before using your GDC
- put information into the GDC accurately
- sensibly use the GDC's ability to work to nine decimal places.

It is very easy to accidentally press the wrong key on your GDC, which could lead to an incorrect answer. If you already have an estimate of what you think the answer will be, this can help you to identify if the answer your GDC gives you looks about right; if it is completely different to what you expected, perhaps you pressed a wrong key and need to enter the calculation again.

Approximation by rounding

Rounding is an idea that many people apply instinctively.

When someone asks you 'how long was that phone call?', you would usually not reply 'nine minutes and thirty-six seconds'. You would probably say 'about ten minutes', rounding your answer to the nearest minute. The 'nearest minute' is a **degree of accuracy**.

When asked to round a number, you will normally be given the degree of accuracy that you need. Some examples are:

- to the nearest yen
- to the nearest 10 cm
- to the nearest millimetre
- to the nearest hundred
- to one decimal place
- to three significant figures.

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To round numbers you can use either a number line or the following rule:

- If the digit to the right of the digit you are rounding is less than five (<5), then the digit being rounded stays the same.
- If the digit to the right of the digit you are rounding is five or more (≥ 5), then the digit being rounded increases by one.

hint

Remember the meaning of the following symbols:
$<$ 'less than'
$\geq$ 'greater than or equal to'
$\approx$ 'approximately equal to'

Worked example 1.4

Q. Use the rule above to round the following numbers:

(a) 1056.68 yen to the nearest yen

(b) 546.21 cm to the nearest 10 cm

(c) 23.35 mm to the nearest mm

(d) 621 317 to the nearest 100.

A. (a) 1056|.68 yen ≈ 1057 yen

Look at the last digit before the decimal point, 6. The digit to its right is also 6. As 6 > 5, we round the digit before the decimal point to 7.

(b) 54|6.21 ≈ 550 cm

The digit we are rounding is 4, which is followed by 6. As 6 > 5 we round 4 up to 5.

(c) 23|.35 mm ≈ 23 mm

The digit we are rounding is 3, and the digit following it (the first digit after the decimal point) is also 3. As 3 < 5 we leave the digit we are rounding unchanged.

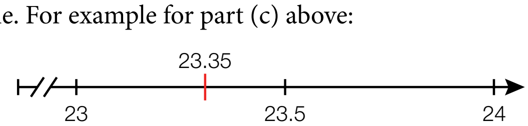
(d) 621 3|17 ≈ 621 300

The digit we are rounding is 3, and it is followed by 1. As 1 < 5 the digit we are rounding is unchanged.

hint

In part (d) of Worked example 1.4, you need to replace the last two digits '1' and '7' by zeros in order to keep the number the correct size. The 3 represents 300, so 317 to the nearest 100 is 300, and therefore '1' and '7' must be replaced by '00'.

You may find it easier to visualise the rounding process using a number line. For example for part (c) above:



23.35 is closer to 23 than to 24, so 23.35 mm = 23 mm to the nearest mm.