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The Riemann Hypothesis for Function Fields

Frobenius Flow and Shift Operators

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To my beautiful wife Jena

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Preface

This book grew out of an attempt to understand the paper [Conn1], in which Alain Connes constructs a beautiful noncommutative space with a view to proving the Riemann hypothesis. That paper is supplemented by Shai Haran's papers [Har2,Har3], which give a similar construction with more details on some of the computations. Connes' proof is explored in Chapter 6, where his method is applied with an aim of proving the Riemann hypothesis for a curve over a finite field (Weil's theorem).

Chapter 5 presents Bombieri's proof [Bom1] of the Riemann hypothesis for curves over a finite field. This chapter is not necessary for Chapter 6, and can be skipped by a reader who is only interested in understanding Connes' approach.

Chapters 1, 2, and 3 provide background. Chapter 1 is an exposition of the theory of valued fields, and in Chapters 2 and 3, we present Tate's thesis [Ta] for curves over a finite field.

There are numerous exercises throughout the book where the reader is asked to work out a detail or explore related material. The exercises that are labelled as 'problems' ask questions that may not have a definite answer.

This book is not primarily about number fields, but occasionally we discuss the connection between number fields and function fields. We have included several diagrams to help the reader create a mental picture of this connection.

The author believes that Connes' approach provides the first truly convincing heuristic argument for the Riemann hypothesis. He also believes that working out this argument for the function field case is the key to getting it to work for the integers. It is therefore not surprising that we do not reach our goal in Chapter 6. This book provides the basis for further research in this direction.

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