1.

ON THE DOUBLE SQUARE REPRESENTATION OF PRIME AND COMPOSITE NUMBERS.

[*York British Association Report*, (1844), Part II. p. 2.]

"THE author first alluded to what had been done by the French mathematicians; and then pointed out the manner in which he thought numbers might be conceived to be composed of squares; and concluded by mentioning some of the advantages which might be expected from this mode of considering them."

2.

A PROBATIONARY LECTURE ON GEOMETRY, DELIVERED BEFORE THE GRESHAM COMMITTEE AND THE MEMBERS OF THE COMMON COUNCIL OF THE CITY OF LONDON, 4 DECEMBER, 1854.

[PRECEDED BY A DEDICATION TO THE MEMBERS OF THE GRESHAM COMMITTEE, MENTIONED BY NAME.]

PREFACE.

To account for the manifold short-comings of the annexed Lecture, it may be excusable and is indeed needful to state the circumstances under which it was written and delivered.

The Author having declared himself a Candidate for the vacant Professorship of Geometry in Gresham College, received a notice of little more than eight and forty hours, that he would be required to deliver a Probationary Lecture on Monday the 4th inst., before the Trustees on the City side of the Gresham Trust.

Matters of pressing importance happening at that moment to absorb his whole attention, he addressed a letter to the Secretary of the Trust, containing an urgent request that he might have the delay of a week for preparation; but his application having been sent too late in the day to obtain a reply, the Author deemed it his duty (not knowing how far his absence might derange the intended proceedings, of the precise nature of which he was unaware) to arm himself with a lecture of some kind, and for better or for worse, to appear to his summons at the appointed place and time. Accordingly, under the necessity of the case, the following pages were commenced and finished at a single sitting of a few hours' duration; the Author being so pressed for time that he had not even enough of it at his disposal to write out a fair copy of the manuscript.

The Lecture, with unimportant exceptions, such as the insertion of the closing paragraph (which was felt but not spoken), the occasional retrenchment of an exuberant expression, or the toning down of an over-florid passage, is printed as it was composed and delivered.

It must not be regarded as a criterion of what the Author could produce, with sufficient leisure, and the usual aids to reflection and research at his command, and still less as a specimen of the kind of lecture which he would consider adapted to a professorial course; but as the hasty outpouring of some of the thoughts lying at the threshold of the subject, and happening at the moment of composition to be most present to his mind. However, with all its imperfections on its head, the Author has deemed that he would be wanting in proper deference to his judges, especially to such of them as were unable to attend in their places on the day of probation, were he to fail to afford them an opportunity of considering it in print. The views put forth and the opinions expressed are, at all events, the result of the Author's own reflections and of the questioning of his own mind, and not of a foray upon the works of the standard writers on the subject.

If his example in printing his Lecture should be followed by any of the numerous body of gentlemen who lectured after him, he will have much gratification in feeling that he has been instrumental in causing the public to participate in the pleasure which he derived from the many excellent discourses which were pronounced on the occasion referred to, and which he hopes to experience again, in listening to the corps of gentlemen from the country, who remain to bring up the reserve of the little army of probationers, on the field day appointed for the 11th instant. He can say unaffectedly and knows that this opinion was shared by many of his fellow-listeners, that the marked variety in the views and manner of treatment adopted by the several lecturers following one another in rapid succession on the same subject, rendered the *concours*, held under the auspices of the Committee and in the presence of the Common Council of the City of London, on the 4th instant, in Gresham Hall, one of the most interesting exhibitions of character and mental differences at which he had ever the good fortune to be present; one, he believes, of most uncommon occurrence, if not altogether unprecedented and unique in this country.

The free and obviously improvised review of his opinions, to which each lecturer was subjected in turn at the hands of those who came after him, threw additional life and spirit into the scene. With scarcely an exception, these light arrows of criticism were untipped with venom and passed sportively to their destination, striking without wounding, or glanced harmlessly off from the impervious shield of good humour interposed to receive them. The usual right of reply under attack must, it is presumed, in this instance be reserved until a fresh vacancy occurs and the same parties re-assemble in the college*.

* The Author will only so far forestall the arrival of the period (*quod longum absit!*) above alluded to, by protesting against the abuse of the word "practical" as employed by an ingenious lecturer who succeeded him at the desk.

To discourse fluently on things of practice is no sufficient evidence in itself of the possession of a practical mind. The first rule of practice is to do all things at the right time and in their proper places, to proportion the means to the ends and the ends to the means, above all to know what is possible, and to confine one's endeavours within the limits of the feasible.

The Author allows and has habitually acted on the principle that for the purpose of *illustrating* lectures on geometry or any other abstract science, the lecturer should lay his hands on the plough, the loom, the forge, the workshop, the mine, the sea, the stars, all things on earth or under heaven, which may help to arouse the attention or interest the imagination of his auditors. But to profess to make the mere applications of a science such as geometry, the staple of the matter to be taught within the walls of the college by the Gresham Lecturer, to undertake to comprise within a course of geometrical lectures systematic *instruction* in mechanics, astronomy and navigation, descriptive geometry, engineering and drawing, the method of interpolation, the theory of toothed wheels, the two kinds of perspective, machinery, mapping, the art of ship-building, rules for cutting the best form of screws, and for enabling the citizens of London to qualify themselves for being their own land-surveyors, is a suggestion which, with all due deference to its propounder, the author regards as one of the wildest and most visionary which ever entered into the mind or issued from the lips of a practical man.

A long life would not suffice to exhaust the circle of the applications of geometry. Sir Thomas Gresham, a true philosopher and man of practical wisdom, ordained that courses of lectures should be delivered on his Foundation, not upon the applications of the sciences but upon the sciences themselves; well knowing that he who has mastered the principles of a science will be capable of making for himself, whenever required, those specific applications of them, which the

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The Author fully agrees with the sentiment expressed by one of the candidates, and emphatically assented to by several members of the Committee, a sentiment which he confidently anticipates will be found to actuate the whole honourable body of the electors, in the decision about to be pronounced by them, in the double and each so sacred character, of Trustees and Judges, which is, that without regard to any other claim than that of capacity and desert, the worthiest candidate may be the one preferred. With his whole heart he is ready to exclaim,

DETUR DIGNIORI.

LECTURE.

MR CHAIRMAN AND GENTLEMEN OF THE GRESHAM COMMITTEE*,

In compliance with your requisition, although upon a very brief and unexpected notice, I have felt it my duty not to shrink from appearing before you this day to deliver a lecture upon the science, of which the chair now stands vacant in Gresham College. A consciousness of want of sufficient opportunity for preparation on my part, and a consideration of the wearisome and laborious duty which you, Gentlemen, have undertaken to perform in listening to a succession of lecturers on the same subject, conspire to impress upon me the importance of condensation and brevity.

I do not propose to tax your patience by entering upon any elaborate discussion of the principles of geometry, its history, its methods, or its applications.

It would be a vain endeavour to seek to convey within the limits of a single lecture any adequate notion of the scope of a science which has engaged the attention and grown up amidst the accumulated labours and meditations of the greatest minds and most profound thinkers of ancient and modern times, which, for the last two or three thousand years, has pursued an almost unbroken course of development and progression, and, still flourishing in all the vigour and freshness of early youth, bids fair to furnish occupation to the reasoning and inventive faculties of mankind for ages yet to come.

I conceive that the purpose for which I have been summoned before you will be best attained and our time most profitably employed, if I confine myself to the suggestion of a

peculiar circumstances of his calling or his opportunities in life may render advisable, and that he who has surprised the citadel will have no difficulty in carrying the outworks.

In writing with reluctance and under a sense of duty, the above remarks, the author begs most distinctly to disclaim the intention of leading it to be inferred that he considers the statement of opposite views as constituting the slightest ground of disqualification in the candidate who gave expression to them. He cannot but surmise that they were hazarded under the exigency of the occasion and in the absence of sufficient time for mature reflection. It has appeared to him however to be not the less necessary on that account, seeing that they were put forth with considerable plausibility, and chime in with some confused notions of what is really useful and practical, which are too prevalent at the present day, and would if carried out be fatal to the cause of sound education, to express his own dissent from them, and to meet them with an immediate refutation. He would be doing pain to his feelings were he not to add that he entertains a high respect for the abilities of the gentleman whose opinions he has felt himself under the necessity of controverting, and that he considers him to occupy a place amongst the foremost rank of those whose election by the Committee would be received with satisfaction by the Public.

* In the place of the Lord Mayor, who was unavoidably absent, Mr Deputy Holt presided.

few general ideas, which will admit of being rendered easily intelligible to any sensible man who, without having made geometry his special study, may be disposed to form some conception of the nature of its subject-matter and the relation which it bears to the other mathematical sciences.

There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order*.

Arithmetic has for its object the properties of number in the abstract. In algebra, viewed as a science of operations, order is the predominating idea. The business of geometry is with the evolution of the properties and relations of Space, or of bodies viewed as existing in space; it is true that the ideas of quantity and order enter largely into the developments of the science, but its proper purpose and foundation, that which confers upon it its distinctive name and character, is the contemplation of the properties of space or of the relations of the parts of space to one another, and only through this, or the equivalent notion of infinite extension, can any approach be made to a just appreciation of its objects.

It is the province of the metaphysician to inquire into the nature of space as it exists in itself, or with relation to the human mind. The less aspiring but more satisfactory business of the geometer is to deal with space as an objective reality, and to view it in its relation to matter, and as the substratum or the condition necessary to the existence of our conception of form.

The first property which strikes the mind in dwelling upon the idea of space is its infinitude, its capacity of boundless extension. If we stretch our thoughts to the very verge of the universe we are still unable to conceive space as come to an end and are constrained to admit the existence of further space beyond.

We may next contemplate space with reference to its modes of extension. We frequently hear of space having three dimensions; that there exist subordinate forms of space in which one or more of these dimensions are wanting; we are all familiar with such forms of speech, with the ideas attaching to the terms solidity, surface, linear magnitude or direction; let us inquire how the notions which they convey may be conceived to arise.

If we imagine a solid figure indefinitely expanded and extended in all directions, we fall back upon the idea of infinite space. If this space be conceived to be subjected to an ideal separation into two parts, distinct but contiguous, the boundary of each such part will give rise to the notion of a superficies, which may be conceived as co-extensive with the space from which it is derived, and like it, infinite in extent. A continuation of this process, that is to say, the dichotomy of superficies in its turn into distinct contiguous parts, gives birth to the notion of an infinite line; a surface limits space, a line limits a surface, and is thus a limitation upon a limitation; now again, conceive a line to separate into two parts and we arrive at the notion of a point, the lowest term in the scale of geometrical being; for here our analysis comes to an end; we have arrived at a limitation of the third order and can go no further; the point admits of no division. Thus it is we become able to attach a distinct meaning to the well known axiom or definition of the old geometers,

* The subject-matter of the notion of abstract order or arrangement is undoubtedly time; thus number, space and time may be said to be the three mathematical categories giving birth to three pure mathematical theories, viz.: arithmetic, the most abstract of all, next tactic or the doctrine of aggregation, and finally geometry, or topic. Each of these again has a double aspect and admits of being pursued in a descriptive and in a quantitative direction.

that "a point is that which hath no parts." Three several times, as we have seen, the process of division, of dual separation may be applied; but we come at last to that which has no parts, and which consequently, or rather by the very force of the term, is insusceptible of further division.

To say that every solid has length, breadth and depth, and therefore that space has three dimensions, is to convey a very inadequate explanation of the fact to be accounted for; for if we consider a spherical or any other body bounded by a continuous surface without angles or edges, we see nothing to indicate the existence in it of three or any other specific number of directions of measurement; it is true that through the idea of quantity, we may compare any solid whatever with a cube of equal magnitude, which of course will possess three definite directions or dimensions of linear admeasurement; but this is at best a very indirect and imperfect mode of arriving at the notion of the property in question, the property of three-foldness, inherent as a quality in the conception of space under the most general and absolute point of view.

Having thus acquired a notion of surfaces and lines in general, it becomes important to limit our attention in the first instance to the study of the simplest forms of each, and here our intuitions evolved by the latent force of early and unremitting observation, experience and induction, present our minds with the plane and sphere, as the elementary forms of surface, and the right line and circle as the corresponding simplest forms of lines.

A plane surface should be always conceived for the purposes of the geometer as extending out indefinitely in all directions; it consists of parts capable of exact superposition each over every other, so that if two portions of a plane be supposed to be brought together they cannot contain a closed hollow between them. A plane may be folded down over itself, and if two planes coincide in three points, they must coincide throughout their whole infinite extent.

Different as a sphere and a plane surface may at first sight appear to be, they possess many properties in common; the most striking difference between them, but which turns out to be comparatively unimportant under a mathematical point of view, consists in the circumstance of the plane being free and unlimited in extent, whereas a sphere is a closed and bounded surface; but in the property of the parts of either being similar *inter se*, and capable of superposition upon one another, there is a perfect resemblance between the sphere and the plane. Nor is it at all necessary to consider the sphere as a result of the idea of the circle, or to define it as Euclid does, as produced by the revolution of a circle about its diameter; we may even form a complete notion of a sphere by regarding it as a simple whole without any express reference to a centre or radii, as a surface containing a solid figure and capable of moving in its own place, without encroaching upon the neighbouring parts of space exterior to itself.

As the plane and sphere are the simplest of all surfaces, so the right line and circle are the simplest of all lines. The right line and circle, like the plane and sphere, are each moveable in their own place, that is, they admit of their parts being shifted upon one another without any absolute change of place in the entire line. There is only one other line in nature, namely the screw line (well known as the helix or Archimedes' screw), which possesses this property of self-similarity, which is the final reason why all the simple mechanical powers exhibit only three sorts of motion, namely, rectilinear, circular, and helical; thus in the lever and toothed wheels circular motion is exemplified; in the pulley and inclined plane, rectilinear motion; and finally helical motion in the screw, such as is used in an ordinary press. I need hardly add that it is the screw of Archimedes which has lent a new power to steam navigation, and which imparts to the rifled barrel its sure and deadly aim.

The foundation of the ancient (indeed it may be said of all) geometry is laid in the contemplation of the properties of figures, capable of being drawn upon a plane, and especially of the simplest of these, namely, the right line and circle.

From the time of Thales, who flourished about 600 years before the birth of Christ, and is reputed to have been the first to bring geometry from the land of the Pharaohs to find a more genial home in Greece, down to the time of Plato, two centuries later, the attention of geometers appears to have been almost exclusively confined to the study of the properties of these simple species of form, and as derived from them, of the sphere and solid figures bounded by plane faces.

The discovery of the conic sections, attributed to Plato, first threw open the higher species of form to the contemplation of geometers*. But for this discovery, which was probably regarded in Plato's time and long after him, as the unprofitable amusement of a speculative brain, the whole course of practical philosophy at the present day, of the science of astronomy, of the theory of projectiles, of the art of navigation, might have run in a different channel; and the greatest discovery that has ever been made in the history of the world, the law of universal gravitation, with its innumerable direct and indirect consequences and applications to every department of human research and industry, might never to this hour have been elicited.

This law, as you are aware, is deduced from the motions of the heavenly bodies in their orbits; no correct system of physical astronomy, no knowledge of the forces binding together the distant parts of the universe was possible, until the form of their orbits had been correctly ascertained by observation.

It is to Kepler, Newton's precursor, that we are indebted for this important information. He it was who discovered that the motion of a planet is not circular nor derived from any combination of circular movements, as was previously supposed to be the case, from a perfection idly supposed to be inherent in that figure, which rendered it alone worthy to image the movements of the heavenly bodies. Kepler discovered that the true form of a planet's orbit is that of an oval perspective projection of a circle, familiar to the geometricians of the Platonic school under the name of an elliptic section of the cone; such also is the general form of the orbits of the moon, of the satellites to the other planets, and in a word of all the bodies in nature revolving about centres of force, subject only to deviations of more or less consequence, arising from disturbing forces for which geometry is perfectly able to account. Thus (as I have said) it was, that the way was laid open to the discovery of this great secret of nature. Little could Plato himself have imagined, when, indulging his instinctive love of the true and beautiful for their own sakes, he entered upon these refined speculations and revelled in a world of his own creation, that he was writing the grammar of the language in which it would be demonstrated in after ages that the pages of the universe are written.

As Plato and Pythagoras before him, the two greatest philosophers of ancient times, have stamped their names upon, and indissolubly associated their memories with the history of the geometry of their period, so the new geometry which has arisen in later days, and achieved still higher triumphs than its elder-born sister, may be said to have taken its origin in the methods invented by Descartes and Pascal, the great philosophical luminaries of modern times. It may be doubted whether Newton could have ever risen to the heights which he attained had not Descartes lived and written before him, and it may be difficult to pronounce the existence of which of the two, Kepler or Descartes, ought to be considered as the more essential link in the order of events prepared by

* Here the Lecturer with the aid of a model showed how the different species of plane conics may be obtained from the dissection of a solid cone.

providence, to furnish the materials to be elaborated by the genius of Newton and to fit it for its lofty appointed work.

But I must not allow myself to be tempted into the facile and seductive path of historical investigation or comparison, which would carry me far beyond the limits which I prescribed to myself at the outset of this discourse, or than I can hope to carry your indulgent attention along with me. For obvious reasons also I think it would be inexpedient to attempt in this place a description of the difference between the spirit and methods of the ancient and those of the modern schools of geometry. I shall prefer to occupy the short remaining period of the lecture with inviting your attention to a distinction which lies deeper in the subject-matter of the science itself, being drawn from its relation to the two leading attributes of space, namely, magnitude and direction. I allude to a distinction, which or the like to which, runs through every branch of mathematical speculation, and has its analogue even in the study of the natural sciences, such as chemistry, botany and anatomy.

When we have attained a certain elevation in our view of the subject, and can look down upon the territory which we have traversed to arrive there, we begin to perceive that geometry resolves itself naturally into two great divisions, geometry of position and geometry of measurement, geometry descriptive* or morphological and geometry quantitative or metrical. The ancients chiefly concerned themselves with the metrical properties of space; the more subtle and essential spirit of the science, however, probably resides in the purely descriptive part. A single proposition selected from each may serve to place the distinction between these two provinces of inquiry in a clearer light.

If we draw any two triangles upon the same base, say for instance along this floor where the wall meets it, terminating respectively in two points, (so chosen that their line of junction shall be parallel to the base line) as for instance to two points in the line running along the cornice of the room, it is easily proved that the two triangles so formed, will be of equal superficial magnitude; this would be true although the apex of one of them were taken anywhere along the actual line of the ceiling, but the other in a prolongation of the cornice stretching out a hundred miles away. Both triangles so obtained would contain the same number of square inches or square feet, although the measure of one round its periphery might be a thousand times greater than that of the other. This is an example of a metrical or quantitative proposition. Again, if we take a triangle and bisect each side and join each bisecting point with the opposite angle, it is a known property of the triangle that these three lines must meet one another, not as three lines taken at hazard would do, cutting out another triangle between them, but in one and the same point. This proposition is partly metrical and partly descriptive; it is descriptive so far as regards the property of the bisecting lines passing through the same point; quantitative in so far as the idea of a line being bisected implies a notion of the relative magnitudes of the equal parts.

Propositions however exist which are purely descriptive; as for instance, the celebrated theorem of Pascal known under the name of the Mystic Hexagram, which is, that if you take two straight lines in a plane, and draw at random other straight lines traversing in a zigzag fashion between them, from A in the first to B in the second, from B in the second to C in the first, from C in the first to D in the second, from D in the second to E in the first, from E in the first to F in the second and finally from F in the second back again to A the starting point in the first, so as to obtain $ABCDEF$ a twisted hexagon, or sort of cat's-cradle figure and if you arrange the six lines so drawn symmetrically in three couples: viz. the 1st and 4th in one couple, the 2nd and 5th in a second couple, the 3rd

* The word "descriptive" is here employed *out* of its technical sense.

and 6th in a third couple; then (no matter how the points ACE have been selected upon one of the given lines, and BDF upon the other) the three points through which these three couples of lines respectively pass, or to which they converge (as the case may be) will lie all in one and the same straight line.

This is a purely descriptive proposition, it refers solely to position, and neither invokes nor involves the idea of magnitude. The existence, I will not say of a class, but of a whole world of truths of this kind, truths undeniably geometrical in their nature, serves to show how imperfect is the definition once generally accepted of geometry (however conformable to the etymology of the word and the early history of the subject), which described it as the science of the measurement of magnitude, in a word as a science of mensuration, which is in fact only one and that a subordinate division of the science. Sciences, true sciences, spring from celestial seeds sown in a mortal soil, they outgrow the restrictions which human shortsightedness seeks to impose upon them, and spread themselves outwards and upwards to the heavens from whence they derive their birth. We may write learnedly upon the history of geometry, upon its origin, growth, and apparent tendencies; but there is that within it which baffles our predictions and sets at nought our calculations as to the uses to which it may hereafter be turned and the form which it may be finally destined to assume; that which, analogous to the vital principle in an organized being, resists the circumscription of language and defies mere verbal definition.

It has been said that to appreciate what virtue and morals mean, men must live virtuous and moral lives. It is equally true, that a knowledge of the objects of science is not to be attained by any scheme of definitions however carefully contrived. He who would know what geometry is, must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied and am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakened and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage "*abeunt studia in mores.*"

I have now only to thank you, Mr Chairman and Gentlemen, for the patient attention which you have accorded to me, and to assure you with perfect sincerity, that if I should have the honour of being selected by you for the permanent occupation of the chair which I this day fill upon trial, I shall not treat the appointment as a sinecure, nor content myself with discharging the mere duties of routine. Far otherwise! if accredited by you to teach publicly a science, the object of my passionate fondness and earliest predilection, to propagate a taste for which would be to me, not merely a labour of duty but of love, I should strive, both in and out of the lecture room, to respond to the intentions of your enlightened and munificent Founder, by imparting freely to all who might approach me for the purpose, advice, encouragement, and sympathy, in their pursuit of mathematical truth, and I should labour with unceasing diligence to evince myself a worthy successor of the many eminent men, who have previously occupied here the chair which it is my ambition to obtain.

As one who has given pledges to the world of an earnest devotion to science, who lays claim to the possession of faculties which would find or create here a fitting theatre for their development, I appeal to your public spirit. I seek, Gentlemen, at your hands to be placed in a position which shall entitle me to take a part in bringing this noble Institution into connection with the great movement of national education now in progress throughout the land, and as a professor in this place, to be permitted to dedicate the past and future labours of my life to the promotion of the general good. The privilege to be useful is the crown of honour which I covet, and which it is in your power to bestow.

3.

NOTE ON SIR JOHN WILSON'S THEOREM.

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 84, 85.]

THE following is probably the best and the briefest mode of deducing Sir John Wilson's Theorem and its cognate Theorems from Fermat's. I can say nothing as to its originality.

p being any prime number, let

$$(x-1)(x-2)(x-3)\dots\{x-(p-1)\} = x^{p-1} + A_1x^{p-2} + A_2x^{p-3} + \&c. + A_{p-1}.$$

Let x successively take the values $1, 2, 3, \dots (p-1)$; then to modulus p, by Fermat's Theorem, we have

$$x^{p-1} + A_{p-1} \equiv 1 + A_{p-1}, \text{ say } A_0,$$

and we derive the $(p-1)$ congruences to modulus p:

$$A_0 + A_1 + A_2 + A_3 \dots\dots\dots\dots + A_{p-2} \equiv 0,$$
$$A_0 + 2^{p-2}A_1 + 2^{p-3}A_2 + 2^{p-4}A_3 \dots + 2A_{p-2} \equiv 0,$$
$$A_0 + 3^{p-2}A_1 + 3^{p-3}A_2 + 3^{p-4}A_3 \dots + 3A_{p-2} \equiv 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$A_0 + (p-1)^{p-2}A_1 + (p-1)^{p-3}A_2 + (p-1)^{p-4}A_3 \dots + (p-1)A_{p-2} \equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0, A_1, A_2, \dots A_{p-2}$$

is $1.2.3\dots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \dots (p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0 \equiv 0, \ A_1 \equiv 0, \dots A_{p-2} \equiv 0 \ [\text{mod. } p].$$

The congruence $A_0 \equiv 0$, that is $1 + 1.2.3\dots(p-1) \equiv 0$ [mod. p], is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3, \dots (p-1)$, are all severally congruent to zero to the modulus p; that is, are all divisible by that number.