THE

FOUNDATIONS OF GEOMETRY.

AN ESSAY

ON THE

FOUNDATIONS OF GEOMETRY

BY

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PREFACE.

THE present work is based on a dissertation submitted at the Fellowship Examination of Trinity College, Cambridge, in the year 1895. Section B of the third chapter is in the main a reprint, with some serious alterations, of an article in *Mind* (New Series, No. 17). The substance of the book has been given in the form of lectures at the Johns Hopkins University, Baltimore, and at Bryn Mawr College, Pennsylvania.

My chief obligation is to Professor Klein. Throughout the first chapter, I have found his "Lectures on non-Euclidean Geometry" an invaluable guide; I have accepted from him the division of Metageometry into three periods, and have found my historical work much lightened by his references to previous writers. In Logic, I have learnt most from Mr Bradley, and next to him, from Sigwart and Dr Bosanquet. On several important points, I have derived useful suggestions from Professor James's "Principles of Psychology."

My thanks are due to Mr G. F. Stout and Mr A. N. Whitehead for kindly reading my proofs, and helping me by many useful criticisms. To Mr Whitehead I owe, also, the inestimable assistance of constant criticism and suggestion throughout the course of construction, especially as regards the philosophical importance of projective Geometry.

HASLEMERE.
May, 1897.

TO

JOHN McTAGGART ELLIS McTAGGART

TO WHOSE DISCOURSE AND FRIENDSHIP IS OWING

THE EXISTENCE OF THIS BOOK.

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