A COURSE IN MATHEMATICAL ANALYSIS

Volume II: Metric and Topological Spaces, Functions of a Vector Variable

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. This second volume goes on to consider metric and topological spaces. Topics such as completeness, compactness and connectedness are developed, with emphasis on their applications to analysis. This leads to the theory of functions of several variables: differentiation is developed in a coordinate free way, while integration (the Riemann integral) is established for functions defined on subsets of Euclidean space. Differential manifolds in Euclidean space are introduced in a final chapter, which includes an account of Lagrange multipliers and a detailed proof of the divergence theorem. Volume III covers complex analysis and the theory of measure and integration.

D. J. H. GARLING is Emeritus Reader in Mathematical Analysis at the University of Cambridge and Fellow of St. John's College, Cambridge. He has fifty years' experience of teaching undergraduate students in most areas of pure mathematics, but particularly in analysis.

A COURSE IN MATHEMATICAL ANALYSIS

Volume II

Metric and Topological Spaces, Functions of a Vector Variable

D. J. H. GARLING
Emeritus Reader in Mathematical Analysis,
University of Cambridge, and
Fellow of St John's College, Cambridge

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

The Edinburgh Building, Cambridge CB2 8RU, UK

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107675322

First published 2013
Reprinted 2014

Printed in the United Kingdom by CPI Group Ltd, Croydon CR0 4YY

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication Data
Garling, D. J. H.
Metric and topological spaces, functions of a vector variable / D. J. H. Garling.
pages cm. – (A course in mathematical analysis; volume 2)
Includes bibliographical references and index.
ISBN 978-1-107-03203-3 (hardback) – ISBN 978-1-107-67532-2 (paperback)
1. Metric spaces 2. Topological spaces. 3. Vector valued functions. I. Title.
QA611.28.G37 2013
514′.325–dc23 2012044992

ISBN 978-1-107-03203-3 Hardback
ISBN 978-1-107-67532-2 Paperback

Contents

Volume II

Introduction

This book is the second volume of a full and detailed course in the elements of real and complex analysis that mathematical undergraduates may expect to meet. Indeed, it was initially based on those parts of analysis that undergraduates at Cambridge University meet, or used to meet, in their first two years. There is however always a temptation to go a bit further, and this is a temptation that I have not resisted. Thus I have included accounts of Baire's category theorem, and the Arzelà–Ascoli theorem, which are taught in the third year, and the Mazur–Ulam theorem, which, as far as I know, has never been taught. As a consequence, there are certain sections that can be omitted on a first reading. These are indicated by asterisks.

Volume I was concerned with analysis on the real line. In Part Three, the analysis is extended to a more general setting. We introduce and consider metric and topological spaces, and normed spaces. In fact, metric and metrizable spaces are sufficient for all subsequent needs, but many of the properties that we investigate are topological properties, and it is well worth understanding what this means. The study of topological spaces can degenerate into the construction of pathological examples; once again, temptation is not resisted, and Section 11.6 contains a collection of these. This section can be omitted at a first reading (and indeed at any subsequent reading). Baire's category theorem is proved in Section 12.6; it is remarkable that a theorem with a rather easy proof can lead to so many strong conclusions, but this is another section that can be omitted at a first reading. The notion of compactness, which is a fundamental topological idea, is studied in some detail. Tychonoff's theorem on the compactness of the product of compact spaces, which involves the axiom of choice, is too hard to include here: a proof is given in Appendix D.

In Part Four, we come back down to earth. The principal concern is the differentiation and integration of functions of several variables. Differentiation is interesting and reasonably straightforward, and we consider functions defined on a normed space; this shows that the results do not depend on any particular choice of coordinate system. Integration is another matter. To begin with it seems that the ideas of Riemann integration developed in Part Two carry over easily to higher dimensions, but serious problems arise as soon as a non-linear change of variables is considered. It is however possible to establish results that suffice in a great number of contexts. For example, the change of variables results are used in Volume III, where we introduce the Lebesgue measure, and the corresponding theory of integration. These results on differentiation and integration are applied in Chapter 19, where we consider subspaces of a Euclidean space which are differential manifolds – subspaces which locally look like Euclidean space.

This volume requires the knowledge of some elementary results in linear algebra; these are described and established in Appendix B.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. All help to develop an understanding of the theory: do them!

I am extremely grateful to Zhuo Min 'Harold' Lim who read the proofs, and found embarrassingly many errors. Any remaining errors are mine alone. Corrections and further comments can be found on a web page on my personal home page at `www.dpmms.cam.ac.uk`.