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Random Walks and Heat Kernels on Graphs

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To my mother, Yvonne Barlow, and in memory of my father, Andrew Barlow.

Contents

Preface

The topic of random walks on graphs is a vast one, and has close connections with many other areas of probability, as well as analysis, geometry, and algebra. In the probabilistic direction, a random walk on a graph is just a reversible or symmetric Markov chain, and many results on random walks on graphs also hold for more general Markov chains. However, pursuing this generalisation too far leads to a loss of concrete interest, and in this text the context will be restricted to random walks on graphs where each vertex has a finite number of neighbours.

Even with these restrictions, there are many topics which this book does not cover – in particular the very active field of relaxation times for finite graphs. This book is mainly concerned with infinite graphs, and in particular those which have polynomial volume growth. The main topic is the relation between geometric properties of the graph and asymptotic properties of the random walk. A particular emphasis is on properties which are stable under minor perturbations of the graph – for example the addition of a number of diagonal edges to the Euclidean lattice $\mathbb{Z}^2$. The precise definition of 'minor perturbation' is given by the concept of a rough isometry, or quasi-isometry. One example of a property which is stable under these perturbations is transience; this stability is proved in Chapter 2 using electrical networks. A considerably harder theorem, which is one of the main results of this book, is that the property of satisfying Gaussian heat kernel bounds is also stable under rough isometries.

The second main theme of this book is deriving bounds on the transition density of the random walk, or the heat kernel, from geometric information on the graph. Once one has these bounds, many properties of the random walk can then be obtained in a straightforward fashion.

Chapter 1 gives the basic definition of graphs and random walks, as well as that of rough isometry. Chapter 2 explores the close connections between

random walks and electrical networks. The key contribution of network theory is to connect the hitting properties of the random walk with the effective resistance between sets. The effective resistance can be bounded using variational principles, and this introduces tools which have no purely probabilistic counterpart.

Chapter 3 introduces some geometric and analytic inequalities which will play a key role in the remainder of the book – two kinds of isoperimetric inequality, Nash inequalities, and Poincaré inequalities. It is shown how the two analytic inequalities (i.e. Nash and Poincaré) can be derived from the geometric information given by the isoperimetric inequalities.

Chapter 4 studies the transition density of the discrete time random walk, or discrete time heat kernel. Two initial upper bounds on this are proved – an 'on-diagonal' upper bound from a Nash inequality, and a long range bound, the Carne–Varopoulos bound, which holds in very great generality. Both these proofs are quite short. To obtain full Gaussian or sub-Gaussian bounds much more work is needed. The second half of Chapter 4 makes the initial steps, and introduces conditions (bounds on exit times from balls, and a 'near-diagonal lower bound') which if satisfied will lead to full Gaussian bounds.

Chapter 5 introduces the continuous time random walk, which is in many respects an easier and more regular object to study than the discrete time random walk. In this chapter it is defined from the discrete time walk and an independent Poisson process – another construction, from the transition densities, is given in the Appendix. It is proved that Gaussian or sub-Gaussian heat kernel bounds hold for the discrete time random walk if and only if they hold for the continuous time walk.

Chapter 6 proves sub-Gaussian or Gaussian bounds for two classes of graphs. The first are 'strongly recurrent' graphs; given the work done in Chapter 4 the results for these come quite easily from volume and resistance estimates. The second class are graphs which satisfy a Poincaré inequality – these include $\mathbb{Z}^d$ and graphs which are roughly isometric to $\mathbb{Z}^d$. The proof of Gaussian bounds for these graphs is based on the tools developed by Nash to study divergence form PDE; these methods also work well in the graph context.

Chapter 7 gives a brief introduction to potential theory. Using the method of balayage, the heat kernel bounds of Chapter 6 lead to a Harnack inequality, and this in turn is used to prove the strong Liouville property (i.e. all positive harmonic functions are constant) for graphs which are roughly isometric to $\mathbb{Z}^d$.

This book is based on a course given at the first Pacific Institute for the Mathematical Sciences (PIMS) summer school in Probability, which was held at UBC in 2004. It is written for first year graduate students, though it is probably too long for a one semester course at most institutions. A much shorter

version of this course was also given in Kyoto in 2005, and I wish to thank both PIMS and the Research Institute for the Mathematical Sciences (RIMS) for their invitations to give these courses.

I circulated a version of these notes several years ago, and I would like to thank those who have read them and given feedback, and in particular, Alain Sznitman and Gerard Ben-Arous, for encouraging me to complete them. I also wish to thank Alain for making available to me some lecture notes of his own on this material.

I have learnt much from a number of previous works in this area – the classical text of Spitzer [Spi], as well as the more recent works [AF, LL, LP, Wo].

I am grateful to those who have sent me corrections, and in particular to Yoshihiro Abe, Alma Sarai Hernandez Torres, Guillermo Martinez Dibene, Jun Misumi, and Zichun Ye.