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Excerpt
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# 1

# Introduction

## 1.1 Graphs and Weighted Graphs

We start with some basic definitions. To be quite formal initially, a graph is a pair $\Gamma = (\mathbb{V}, E)$. Here $\mathbb{V}$ is a finite or countably infinite set, and $E$ is a subset of $\mathscr{P}_2(\mathbb{V}) = \{\text{subsets } A \text{ of } \mathbb{V} \text{ with two elements}\}$. Given a set $A$ we write $|A|$ for the number of elements of $A$. The elements of $\mathbb{V}$ are called *vertices*, and the elements of $E$ *edges*. In practice, we think of the elements of $\mathbb{V}$ as points (often embedded in some space such as $\mathbb{R}^d$), and the edges as lines between the vertices.

Now for some general definitions.

(1) We write $x \sim y$ to mean $\{x, y\} \in E$, and say that $y$ is a *neighbour* of $x$. Since all edges have two distinct elements, we have not allowed edges between a point $x$ and itself. Also, we do not allow multiple edges between points, though we could; anything which can be done with multiple edges can be done better with weights, which will be introduced shortly.

(2) A path $\gamma$ in $\Gamma$ is a sequence $x_0, x_1, \ldots, x_n$ with $x_{i-1} \sim x_i$ for $1 \le i \le n$. The length of a path $\gamma$ is the number of edges in $\gamma$. A path $\gamma = (x_0, \ldots, x_n)$ is a *cycle* or *loop* if $x_0 = x_n$. A path is *self-avoiding* if the vertices in the path are distinct.

(3) We define $d(x, y)$ to be the length $n$ of the shortest path $x = x_0, x_1, \ldots, x_n = y$. If there is no such path then we set $d(x, y) = \infty$. (This is the *graph* or *chemical metric* on $\Gamma$.) We also write for $x \in \mathbb{V}$, $A \subset \mathbb{V}$,

$$d(x, A) = \min\{d(x, y) : y \in A\}.$$

(4) $\Gamma$ is *connected* if $d(x, y) < \infty$ for all $x, y$.

(5) $\Gamma$ is *locally finite* if $N(x) = \{y : y \sim x\}$ is finite for each $x \in \mathbb{V}$, that is, every vertex has a finite number of neighbours. $\Gamma$ has *bounded geometry* if $\sup_x |N(x)| < \infty$.
(6) Define balls in $\Gamma$ by

$$B(x,r) = \{y : d(x,y) \le r\}, \quad x \in \mathbb{V}, \quad r \in [0,\infty).$$

Note that while the metric $d(x,\cdot)$ is integer valued, we allow $r$ here to be real. Of course $B(x,r) = B(x,\lfloor r \rfloor)$.
(7) We define the *exterior boundary of* $A$ by

$$\partial A = \{y \in A^c : \text{ there exists } x \in A \text{ with } x \sim y\}.$$

Set also

$$\begin{aligned} \partial_i A &= \partial(A^c) = \{y \in A : \text{ there exists } x \in A^c \text{ with } x \sim y\}, \\ \overline{A} &= A \cup \partial A, \\ A^o &= A - \partial_i A. \end{aligned}$$

We use the notation $A_n \uparrow\uparrow \mathbb{V}$ to mean that $A_n$ is an increasing sequence of finite sets with $\cup_n A_n = \mathbb{V}$.

**Notation** We write $c_i$, $C_i$ for (strictly) positive constants, which will remain fixed in each argument. In some proofs we will write $c, c'$ etc. for positive constants, which may change from line to line. Given functions $f, g : A \to \mathbb{R}$ we write $f \asymp g$ to mean that there exists $c \ge 1$ such that

$$c^{-1} f(x) \le g(x) \le c f(x) \quad \text{for all } x \in A.$$

We write $\mathbf{1}$ for the function $f$ which is identically 1, and $\mathbf{1}_A$ for the indicator function (or characteristic function) of the set $A$. We write $\mathbf{1}_x$ for $\mathbf{1}_{\{x\}}$.

From now on we will always assume:

$$\Gamma \text{ is locally finite,} \tag{H1}$$

$$\Gamma \text{ is connected;} \tag{H2}$$

and in addition from time to time we will assume:

$$\Gamma \text{ has bounded geometry.} \tag{H3}$$

We will treat *weighted graphs*.

**Definition 1.1** We assume there exist weights (also called *conductances*) $\mu_{xy}$, $x, y \in \mathbb{V}$ satisfying:

(i) $\mu_{xy} = \mu_{yx}$,
(ii) $\mu_{xy} \geq 0$ for all $x, y \in \mathbb{V}$,
(iii) if $x \neq y$ then $\mu_{xy} > 0$ if and only if $x \sim y$.

We call $(\Gamma, \mu)$ a *weighted graph*. Note that condition (iii) allows us to have $\mu_{xx} > 0$. Since $E = \big\{\{x, y\} : x \neq y, \mu_{xy} > 0\big\}$, we can recover the set of edges from $\mu$.

The *natural weights* on $\Gamma$ are given by

$$\mu_{xy} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

Whenever we discuss a graph without explicit mention of weights, we will assume we are using the natural weights.

Let $\mu_x = \mu(x) = \sum_y \mu_{xy}$, and extend $\mu$ to a measure on $\mathbb{V}$ by setting

$$\mu(A) = \sum_{x \in A} \mu_x. \tag{1.1}$$

Since $\Gamma$ is locally finite we have:

(i) $|B(x, r)| < \infty$ for any $x$ and $r$,
(ii) $\mu(A) < \infty$ for any finite set $A$.

We will sometimes need additional conditions on the weights.

**Definition 1.2** We say $(\Gamma, \mu)$ has *bounded weights* if there exists $C_1 < \infty$ such that $C_1^{-1} \leq \mu_e \leq C_1$ for all $e \in E$, and $\mu_{xx} \leq C_1$ for all $x \in \mathbb{V}$. We say $(\Gamma, \mu)$ has *controlled weights* if there exists $C_2 < \infty$ such that

$$\frac{\mu_{xy}}{\mu_x} \geq \frac{1}{C_2} \quad \text{whenever } x \sim y.$$

We introduce the conditions:

$$(\Gamma, \mu) \text{ has bounded weights,} \tag{H4}$$

$$(\Gamma, \mu) \text{ has controlled weights.} \tag{H5}$$

Controlled weights is called 'the $p_0$ condition' in [GT1, GT2]; note that it holds if (H3) and (H4) hold. While a little less obvious than (H4), it turns out that controlled weights is the 'right' condition in many circumstances. As an example of its use, we show how it gives bounds on both the cardinality and the measure of a ball.

**Lemma 1.3** *Suppose* $(\Gamma, \mu)$ *satisfies (H5) with constant* $C_2$. *Then (H3) holds and, for* $n \geq 0$, $|B(x,n)| \leq 2C_2^n$, $\mu(B(x,n)) \leq 2C_2^n \mu_x$.

*Proof* Since $\mu_{xy} \geq \mu_x / C_2$ if $x \sim y$, we have $|N(x)| \leq C_2$ for all $x$ and thus (H3) holds. Further, we have $C_2 \geq 2$ unless $|\mathbb{V}| \leq 2$. Then writing $S(x,n) = \{y : d(x,y) = n\}$, we have $|S(x,n)| \leq C_2 |S(x,n-1)|$, and so by induction $|B(x,n)| \leq (C_2^{n+1} - 1)/(C_2 - 1) \leq 2C_2^n$. Also

$$\mu(N(x)) = \sum_{y \sim x} \mu_y \leq C_2 \sum_{y \sim x} \mu_{xy} = C_2 \mu_x.$$

Hence again by induction $\mu(B(x,n)) \leq 2C_2^n \mu_x$. □

**Examples 1.4** (1) The Euclidean lattice $\mathbb{Z}^d$. Here $\mathbb{V} = \mathbb{Z}^d$, and $x \sim y$ if and only if $|x - y| = 1$. Strictly speaking we should denote this graph by $(\mathbb{Z}^d, E_d)$, where $E_d$ is the set of edges defined above.

(2) The $d$-ary tree. ('Binary' if $d = 2$.) This is the unique infinite graph with $|N(x)| \equiv d + 1$, and with no cycles.

(3) We will also consider the 'rooted binary tree' $\mathbb{B}$. Let $\mathbb{B}_0 = \{o\}$, and for $n \geq 1$ let $\mathbb{B}_n = \{0,1\}^n$. Then the vertex set is given by $\mathbb{B} = \cup_{n=0}^{\infty} \mathbb{B}_n$. For $x = (x_1, \dots, x_n) \in \mathbb{B}_n$ with $n \geq 2$ set $a(x) = (x_1, \dots, x_{n-1})$ – we call $a(x)$ the *ancestor of* $x$. (We set $a(z) = o$ for $z \in \mathbb{B}_1$.) Then the edge set of the rooted binary tree is given by

$$E(\mathbb{B}) = \Big\{ \{x, a(x)\} : x \in \mathbb{B} - \mathbb{B}_0 \Big\}.$$

(4) The *canopy tree* is a subgraph of $\mathbb{B}$ defined as follows. Let

$$\mathbb{U} = \{o\} \cup \{x = (x_1, \dots, x_n) \in \mathbb{B} : x_1 = \cdots = x_n = 0\}.$$

Define $f : \mathbb{B} \to \mathbb{B}$ by taking $f(x)$ to be the point in $\mathbb{U}$ closest to $x$, and set $\mathbb{B}_{\mathrm{Can}} = \{x \in \mathbb{B} : d(x, f(x)) \leq d(o, f(x))\}$. Then the canopy tree is the subgraph of $(\mathbb{B}, E(\mathbb{B}))$ generated by $\mathbb{B}_{\mathrm{Can}}$. The canopy tree has exponential volume growth, but only one self-avoiding path from $o$ to infinity.

(5) Let $\mathscr{G}$ be a finitely generated group, and let $\Lambda = \{g_1, \dots, g_n\}$ be a set of generators, not necessarily minimal. Write $\Lambda^* = \{g_1, \dots, g_n, g_1^{-1}, \dots, g_n^{-1}\}$. Let $\mathbb{V} = \mathscr{G}$ and let $\{g, h\} \in E$ if and only if $g^{-1}h \in \Lambda^*$. Then $\Gamma = (\mathbb{V}, E)$ is the *Cayley graph* of the group $\mathscr{G}$ with generators $\Lambda$.

$\mathbb{Z}^d$ is the Cayley graph of the group $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, with generators $g_k$, $1 \leq k \leq d$; here $g_k$ has 1 in the $k$th place and is zero elsewhere. Since different sets of generators will give rise to different Cayley graphs, in general a group $\mathscr{G}$ will have many different Cayley graphs. The $d$-ary tree is the Cayley graph of the group generated by $\Lambda = \{x_1, x_2, \dots, x_{d+1}\}$ with the relations $x_1^2 = \cdots = x_{d+1}^2 = 1$. Since the ternary tree is also the Cayley graph of the free group with two generators, two distinct (non-isomorphic) groups can have the same Cayley graph.

(6) Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+ = \{0, 1, \dots\}$, but with weights $\mu^{(\alpha)}_{n,n+1} = \alpha^n$. This satisfies (H1)–(H3) and (H5), but (H4) fails unless $\alpha = 1$. If $\alpha < 1$ then $(\mathbb{Z}_+, \mu^{(\alpha)})$ is an infinite graph, but $\mu^{(\alpha)}(\mathbb{Z}_+) < \infty$.

(7) Given a weighted graph $(\mathbb{V}, E, \mu)$, the *lazy* weighted graph $(\mathbb{V}, E, \mu^{(L)})$ is defined by setting $\mu^{(L)}_{xy} = \mu_{xy}$ if $y \neq x$, and

$$\mu^{(L)}_{xx} = \sum_{y \neq x} \mu_{xy}.$$

**Definition 1.5** We introduce the following operations on graphs or weighted graphs. Let $(\Gamma, \mu)$ be a weighted graph.

(1) *Subgraphs.* Let $H \subset \mathbb{V}$. Then the *subgraph induced by H* is the (weighted) graph with vertex set $H$, edge set

$$E_H = \big\{\{x, y\} \in E : x, y \in H\big\},$$

and weights (if appropriate) given by

$$\mu^H_{xy} = \mu_{xy}, \quad x, y \in H.$$

We define $\mu^H_x = \sum_{y \in H} \mu^H_{xy}$; note that $\mu^H_x \le \mu_x$. Clearly $(H, E_H)$ need not be connected, even if $\Gamma$ is.

(2) *Collapse of a subset to a point.* Let $A \subset \mathbb{V}$. The graph obtained by collapsing $A$ to a point $a$ (where $a \notin \mathbb{V}$) is the graph $\Gamma' = (\mathbb{V}', E')$ given by

$$\mathbb{V}' = (\mathbb{V} - A) \cup \{a\},$$
$$E' = \big\{\{x, y\} \in E : x, y \in \mathbb{V} - A\big\} \cup \big\{\{x, a\} : x \in \partial A\big\}.$$

We define weights on $(\mathbb{V}', E')$ by setting $\mu'_{xy} = \mu_{xy}$ if $x, y \in \mathbb{V} - A$, and

$$\mu'_{xa} = \sum_{b \in A} \mu_{xb}, \quad x \in \mathbb{V} - A.$$

Note that $\mu'_{xa} \le \mu_x < \infty$. In general this graph need not be locally finite, but will be if $|\partial A| < \infty$. It is easy to check that if $\Gamma$ is connected then so is $\Gamma'$.

(3) *Join of two graphs.* Let $\Gamma_i = (\mathbb{V}_i, E_i)$, $i = 1, 2$ be two graphs. (We regard $\mathbb{V}_1, \mathbb{V}_2$ as being distinct sets.) If $x_i \in \mathbb{V}_i$, the *join of* $\Gamma_1$ *and* $\Gamma_2$ *at* $x_1, x_2$ is the graph $(\mathbb{V}, E)$ given by

$$\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2, \qquad E = E_1 \cup E_2 \cup \big\{\{x_1, x_2\}\big\}.$$

We define weights on $(\mathbb{V}, E)$ by keeping the weights on $\Gamma_1$ and $\Gamma_2$, and giving the new edge $\{x_1, x_2\}$ weight 1.

(4) *Graph products.* Let $\Gamma_i$ be graphs. There are two natural products, called the *Cartesian product* (denoted $\Gamma_1 \,\square\, \Gamma_2$) and the *direct* or *tensor product* $\Gamma_1 \times \Gamma_2$. In both cases we take the vertex set of the product to be $\mathbb{V}_1 \times \mathbb{V}_2 = \{(x_1, x_2) : x_1 \in \mathbb{V}_1, x_2 \in \mathbb{V}_2\}$. For the Cartesian product we define

$$E_{\square} = \big\{\{(x, y_1), (x, y_2)\} : x \in \mathbb{V}_1, \{y_1, y_2\} \in E_2\big\} \\ \cup \big\{\{(x_1, y), (x_2, y)\} : \{x_1, x_2\} \in E_1, y \in \mathbb{V}_2\big\},$$

and for the tensor product

$$E_{\times} = \big\{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E_1, \{y_1, y_2\} \in E_2\big\}.$$

Note that $\mathbb{Z}^2 = \mathbb{Z} \,\square\, \mathbb{Z}$. If $\mu^{(i)}$ are weights on $\Gamma_i$ then we define weights on the product by

$$\mu^{\square}_{(x,y_1),(x,y_2)} = \mu^{(2)}_{y_1,y_2}, \quad \mu^{\square}_{(x_1,y),(x_2,y)} = \mu^{(1)}_{x_1,x_2}, \\ \mu^{\times}_{(x_1,y_1),(x_2,y_2)} = \mu^{(1)}_{x_1,x_2}\mu^{(2)}_{y_1,y_2}.$$

## 1.2 Random Walks on a Weighted Graph

Set

$$\mathscr{P}(x, y) = \frac{\mu_{xy}}{\mu_x}. \tag{1.2}$$

Let $X$ be the discrete time Markov chain $X = (X_n, n \geq 0, \mathbb{P}^x, x \in \mathbb{V})$ with transition matrix $(\mathscr{P}(x, y))$, defined on a space $(\Omega, \mathscr{F})$ – see Appendix A.2. Here $\mathbb{P}^x$ is the law of the chain with $X_0 = x$, and the transition probabilities are given by

$$\mathbb{P}^z(X_{n+1} = y | X_n = x) = \mathscr{P}(x, y).$$

We call $X$ the *simple random walk* (SRW) on $(\Gamma, \mu)$ and $(\mathscr{P}(x, y))$ the *transition matrix* of $X$. At each time step $X$ moves from a vertex $x$ to a neighbour $y$ with probability proportional to $\mu_{xy}$. Since we can have $\mu_{xx} > 0$ we do in general allow $X$ to remain at a vertex $x$ with positive probability. If $\nu$

is a probability measure on $\mathbb{V}$ we write $\mathbb{P}^\nu$ for the law of $X$ started with distribution $\nu$.

Note that $(\mathscr{P}(x, y))$ is *$\mu$-symmetric*, that is:

$$\mu_x \mathscr{P}(x, y) = \mu_y \mathscr{P}(y, x) = \mu_{xy}. \tag{1.3}$$

Set

$$\mathscr{P}_n(x, y) = \mathbb{P}^x(X_n = y).$$

The condition (1.3) implies that $X$ is reversible.

**Lemma 1.6** *Let $x_0, x_1, \ldots, x_n \in \mathbb{V}$. Then*

$$\begin{aligned}\mu_{x_0} \mathbb{P}^{x_0}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) \\ = \mu_{x_n} \mathbb{P}^{x_n}(X_0 = x_n, X_1 = x_{n-1}, \ldots, X_n = x_0).\end{aligned}$$

*Proof* Using the $\mu$-symmetry of $\mathscr{P}$ it is easy to check that both sides equal $\prod_{i=1}^n \mu_{x_{i-1}, x_i}$. □

**Definition 1.7** Let $A \subset \mathbb{V}$. Define the hitting times

$$\begin{aligned}T_A &= \min\{n \geq 0 : X_n \in A\}, \\ T_A^+ &= \min\{n \geq 1 : X_n \in A\}.\end{aligned}$$

Here we adopt the convention that $\min \varnothing = +\infty$, so that $T_A = \infty$ if and only if $X$ never hits $A$. Note also that if $X_0 \notin A$ then $T_A = T_A^+$. Write $T_x = T_{\{x\}}$ and

$$\tau_A = T_{A^c} = \min\{n \geq 0 : X_n \notin A\} \tag{1.4}$$

for the time of the first exit from $A$. If $A, B \subset \mathbb{V}$ we will write $\{T_A < T_B\}$ for the event that could more precisely be denoted $\{T_A < T_B, T_A < \infty\}$. If $A$, $B$ are disjoint then the sample space $\Omega$ can be written as the disjoint union

$$\begin{aligned}\Omega = \{T_A < T_B, T_A < \infty\} \cup \{T_B < T_A, T_B < \infty\} \\ \cup \{T_A = T_B = \infty\} \cup \{T_A = T_B < \infty\}.\end{aligned}$$

The following theorem is proved in most first courses on Markov chains – see for example [Dur, KSK, Nor], or Appendix A.2.

**Theorem 1.8** *Let $\Gamma$ be connected, locally finite, infinite.*
*(T) The following five conditions are equivalent:*

*(a) There exists* $x \in \mathbb{V}$ *such that* $\mathbb{P}^x(T_x^+ < \infty) < 1$.
*(b) For all* $x \in \mathbb{V}$, $\mathbb{P}^x(T_x^+ < \infty) < 1$.
*(c) For all* $x \in \mathbb{V}$,

$$\sum_{n=0}^{\infty} \mathscr{P}_n(x,x) < \infty.$$

*(d) For all* $x, y \in \mathbb{V}$ *with* $x \neq y$ *either* $\mathbb{P}^x(T_y < \infty) < 1$ *or* $\mathbb{P}^y(T_x < \infty) < 1$.
*(e) For all* $x, y \in \mathbb{V}$,

$$\mathbb{P}^x(X \text{ hits } y \text{ only finitely often}) = \mathbb{P}^x\Big(\sum_{n=0}^{\infty} \mathbf{1}_{(X_n=y)} < \infty\Big) = 1.$$

*(R) The following five conditions are equivalent:*

*(a) There exists* $x \in \mathbb{V}$ *such that* $\mathbb{P}^x(T_x^+ < \infty) = 1$.
*(b) For all* $x \in \mathbb{V}$, $\mathbb{P}^x(T_x^+ < \infty) = 1$.
*(c) For all* $x \in \mathbb{V}$,

$$\sum_{n=0}^{\infty} \mathscr{P}_n(x,x) = \infty.$$

*(d) For all* $x, y \in \mathbb{V}$, $\mathbb{P}^x(T_y < \infty) = 1$.
*(e) For all* $x, y \in \mathbb{V}$,

$$\mathbb{P}^x(X \text{ hits } y \text{ infinitely often}) = \mathbb{P}^x\Big(\sum_{n=0}^{\infty} \mathbf{1}_{(X_n=y)} = \infty\Big) = 1.$$

**Definition 1.9** If (T)(a)–(e) of Theorem 1.8 hold we say $\Gamma$ (or $X$) is *transient*; otherwise (R)(a)–(e) of Theorem 1.8 hold and we say $\Gamma$ or $X$ is *recurrent*. The *type problem* for a graph $\Gamma$ is to determine whether it is transient or recurrent. A special case is given by the Cayley graphs of groups, and is sometimes called *Kesten's problem*: which groups have recurrent Cayley graphs? In this case, recall that the Cayley graph depends on both the group $\mathscr{G}$ and the set of generators $\Lambda$.

Let us start with the Euclidean lattices. Polya [Pol] proved the following in 1921.

**Theorem 1.10** $\mathbb{Z}^d$ *is recurrent if* $d \leq 2$ *and transient if* $d \geq 3$.

Polya used the fact that $X_n$ is a sum of independent random variables, and that therefore the Fourier transform of $\mathscr{P}_n(0,x)$ is a product. Inverting the Fourier transform, he deduced that $\mathscr{P}_{2n}(0,0) \sim c_d n^{-d/2}$, and hence proved his theorem.

Most modern textbooks give a combinatorial proof. For $d = 1$ we have

$$\mathscr{P}_{2n}(0,0) = 2^{-2n}\binom{2n}{n} \sim \frac{1}{(\pi n)^{1/2}};$$

this is easy using Stirling's formula. For $d = 2$ we have

$$\mathscr{P}_{2n}((0,0),(0,0)) = 4^{-2n}\sum_{k=0}^{n}\binom{2n}{k}^2 \sim \frac{1}{\pi n},$$

after a bit more work. (Or we can use the trick of writing $Z_n = (\frac{1}{2}(X_n + Y_n), \frac{1}{2}(X_n - Y_n))$, where $X$ and $Y$ are independent SRW on $\mathbb{Z}$.) In general we obtain $c_d n^{-d/2}$, so the series converges when $d \geq 3$. This expression is to be expected: by the local limit theorem (a variant of the central limit theorem), we have that the distribution of $X_n$ is approximately the multivariate Gaussian $N(0, n^{1/2}d^{-1/2}I_d)$. (Of course the fact that, after rescaling, the SRW $X$ on $\mathbb{Z}^d$ with $d \geq 3$ converges weakly to a transient process does not prove that $X$ is transient, since transience/recurrence is not preserved by weak convergence.)

The advantage of the combinatorial argument is that it is elementary. On the other hand, the details are a bit messy for $d \geq 3$. More significantly, the technique is not robust. Consider the following three examples:

(1) The SRW on the hexagonal lattice in $\mathbb{R}^2$.
(2) The SRW on a graph derived from a Penrose tiling.
(3) The graph $(\Gamma, \mu)$ where $\Gamma = \mathbb{Z}^d$, and the weights $\mu$ satisfy $\mu_{xy} \in [c^{-1}, c]$.

**Example** (1) could be handled by the combinatorial method, but the details would be awkward, since one now has to count six kinds of steps. Also it is plainly a nuisance to have to give a new argument for each new lattice. Polya's method does work for this example, but both methods look hopeless for (2) or (3), since they rely on having an exact expression for $\mathscr{P}_n(x,x)$ or its transform. (For the convergence of SRW on a Penrose tiling to Brownian motion see [T3, BT].)

We will be interested in how the geometry of $\Gamma$ is related to the long run behaviour of $X$. As far as possible we want techniques which are *stable* under various perturbations of the graph.

**Definition 1.11** Let $P$ be some property of a weighted graph $(\Gamma, \mu)$, or the SRW $X$ on it. $P$ is *stable under bounded perturbation of weights (weight stable)* if whenever $(\Gamma, \mu)$ satisfies $P$, and $\mu'$ are weights on $\Gamma$ such that

$$c^{-1}\mu_{xy} \leq \mu'_{xy} \leq c\mu_{xy}, \quad x, y \in \mathbb{V},$$

then $(\Gamma, \mu')$ satisfies $P$. (We say the weights $\mu$ and $\mu'$ are *equivalent*.)

**Definition 1.12** Let $(X_i, d_i)$, $i = 1, 2$ be metric spaces. A map $\varphi : X_1 \to X_2$ is a *rough isometry* if there exist constants $C_1$, $C_2$ such that

$$C_1^{-1}(d_1(x, y) - C_2) \le d_2(\varphi(x), \varphi(y)) \le C_1(d_1(x, y) + C_2), \tag{1.5}$$

$$\bigcup_{x \in X_1} B_{d_2}(\varphi(x), C_2) = X_2. \tag{1.6}$$

If there exists a rough isometry between two spaces they are said to be *roughly isometric*. (One can check that this is an equivalence relation.)

This concept was introduced for groups by Gromov [Grom1, Grom2] under the name *quasi-isometry*, and in the context of manifolds by Kanai [Kan1, Kan2]. A rough isometry between two spaces implies that the two spaces have the same large scale structure. However, to get any useful consequences of two spaces being roughly isometric one also needs some kind of local regularity. This is usually done by considering rough isometries within a family of spaces satisfying some fixed local regularity condition. (For example, Kanai assumed the manifolds had bounded geometry: that is, that the Ricci curvature was bounded below by a constant.) Sometimes one has to be careful not to forget such 'hidden' side conditions.

**Example** $\mathbb{Z}^d$, $\mathbb{R}^d$, and $[0, 1] \times \mathbb{R}^d$ are all roughly isometric.

The following will be proved in Chapter 2 – see Proposition 2.59.

**Proposition 1.13** *Let $\mathcal{G}$ be a finitely generated infinite group, and $\Lambda$, $\Lambda'$ be two sets of generators. Let $\Gamma$, $\Gamma'$ be the associated Cayley graphs. Then $\Gamma$ and $\Gamma'$ are roughly isometric.*

For rough isometries of weighted graphs, the natural additional regularity condition is to require that both graphs have controlled weights. Using Lemma 1.3 and the condition (1.7) below this allows one to relate the measures of balls in the two graphs.

**Definition 1.14** Let $(\Gamma_i, \mu_i)$, $i = 1, 2$ be weighted graphs satisfying (H5). A map $\varphi : \mathbb{V}_1 \to \mathbb{V}_2$ is a *rough isometry* (between $(\Gamma_1, \mu_1)$ and $(\Gamma_2, \mu_2)$) if:

(1) $\varphi$ is a rough isometry between the metric spaces $(\mathbb{V}_1, d_{\Gamma_1})$ and $(\mathbb{V}_2, d_{\Gamma_2})$ (with constants $C_1$ and $C_2$);
(2) there exists $C_3 < \infty$ such that for all $x \in \mathbb{V}_1$

$$C_3^{-1}\mu_1(x) \le \mu_2(\varphi(x)) \le C_3\mu_1(x). \tag{1.7}$$