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Paul Fannon Vesna Kadelburg Ben Woolley and Stephen Ward
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1 COUNTING PRINCIPLES

WHAT YOU NEED TO KNOW

- The product and addition principles:
  - The number of ways in which both event A **and** event B occur is the product of the number of outcomes for each event.
  - The number of ways in which either event A **or** event B occurs is the sum of the number of outcomes for each event (if A and B are mutually exclusive).
- The number of ways of arranging (permuting) n different objects is $n!$.
- The number of ways of choosing r objects from n:
  - is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ when the order **does not** matter
  - is ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ when the order **does** matter.
- When arranging objects:
  - If a number of objects have to be kept together, treat them as a single cluster. Arrange the whole group including this cluster and then arrange the objects within the cluster.
  - If two objects have to be kept apart, subtract the number of ways of arranging the whole group with the two objects together from the total number of possible permutations.
  - If there is a constraint, first put the constrained objects into their positions and then count the number of ways of arranging the unconstrained objects.
- Counting principles can be helpful when calculating probabilities:

$$\mathrm{P}(A) = \frac{\text{number of ways } A \text{ occurs}}{\text{total number of ways}}$$

EXAM TIPS AND COMMON ERRORS

- Deciding whether the order of the objects matters can be tricky. Choosing committees or teams and dealing cards are common situations where order **does not** matter. Forming words or numbers and the results in races are common situations where order **does** matter.
- When a constraint is added, the number of possible combinations should decrease. You should always compare your answer with the unconstrained number of possible combinations (which might have been asked for in an earlier part of the question).
- Make sure you know whether the question is asking for a number of arrangements or a probability.

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1.1 ARRANGING OBJECTS

WORKED EXAMPLE 1.1

Mr and Mrs Singh and their four children line up for a family photograph.

(a) In how many different ways can they do this?
(b) How many ways are there if Mr and Mrs Singh are at opposite ends of the line?
(c) How many ways are there if Mr and Mrs Singh have to be next to each other?
(d) What is the probability that Mr and Mrs Singh are not next to each other?

(a) 6 people can be arranged in $6! = 720$ ways.

(b) Either:

Mr Singh, _ , _ , _ , _ , Mrs Singh

Or:

Mrs Singh, _ , _ , _ , _ , Mr Singh

Consider separately the two cases that result from the constraint.

In each case there are 4! ways of arranging the children.

So there are $4! + 4! = 48$ ways in total.

Apply the addition principle to one case OR the other.

Always check that your answer makes sense. The answer to this part must be less than the answer to part (a).

(c) With Mr and Mrs Singh next to each other, there are 5! ways.

Mr and Mrs Singh can be arranged in 2! ways.

Start by treating Mr and Mrs Singh as one cluster, so we have a total of 5 objects to arrange.

So there are $5! \times 2! = 240$ ways in total.

Apply the product principle, as we have Mr and Mrs Singh next to each other AND they can arrange themselves in 2! ways.

(d) The number of permutations where Mr and Mrs Singh are not next to each other is: $720 - 240 = 480$.

P(Mr and Mrs Singh not next to each other) = $\frac{480}{720} = \frac{2}{3}$.

From part (c), we know there are 240 permutations where Mr and Mrs Singh are next to each other. We need to consider the permutations where they are **not** next to each other.

It is always a good idea to look for links between parts of a question.

Practice questions 1.1

1. The letters of the word 'COUNTERS' are to be arranged.
(a) In how many ways can this be done?
(b) In how many ways can this be done if the 'word' begins with a C?
(c) In how many ways can this be done if the O and the U are separated?
(d) What is the probability that the arrangement contains the word 'COUNT'?

2. How many arrangements of the word 'PICTURE' have all the vowels together?

3. The letters of the word 'TABLE' are to be arranged.
(a) How many different arrangements are possible?
(b) If one of the arrangements is selected at random, what is the probability that the vowels are separated?

4. How many arrangements of the word 'INCLUDE' start and end with a vowel?

5. Six boys, Andrew, Brian, Colin, Daniel, Eddie and Fred, line up for a photo.
(a) How many possible arrangements are there?
(b) How many possible arrangements are there in which:
  (i) Andrew is at one end of the line?
  (ii) Andrew is not at either end?
  (iii) Andrew is at the left end of the line or Fred is at the right end, or both?

Brian, Colin and Daniel are brothers.
(c) In a random arrangement of the six boys, what is the probability that:
  (i) all three brothers sit together?
  (ii) there are brothers on both ends of the line?

6. In a random arrangement of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, what is the probability that the odd numbers will all be separated?

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1.2 CHOOSING FROM GROUPS WHEN ORDER DOES NOT MATTER

WORKED EXAMPLE 1.2

A team of 10 is chosen from a class of 14 girls and 12 boys.

(a) How many of the possible selections contain the same number of girls and boys?
(b) In how many ways can the team be selected if it must contain at least two boys?

(a) 5 girls can be chosen in $\binom{14}{5}$ ways.

5 boys can be chosen in $\binom{12}{5}$ ways.

So number of selections is $\binom{14}{5}\binom{12}{5} = 1585584$

Apply the product principle as we are choosing 5 girls AND 5 boys.

(b) There are $\binom{12}{0}\binom{14}{10}$ ways of choosing 0 boys.

There are $\binom{12}{1}\binom{14}{9}$ ways of choosing 1 boy.

So the number of ways with at least 2 boys is

$$\binom{26}{10} - \left[\binom{12}{0}\binom{14}{10} + \binom{12}{1}\binom{14}{9}\right] = 5286710$$

The phrase 'at least' (or 'at most') is often an indicator that you should look at the opposite situation and subtract from the total.

We are not interested in a selection with 0 boys (therefore 10 girls) or a selection with only 1 boy (therefore 9 girls). We count these and subtract from the total.

Practice questions 1.2

7. A game uses a pack of 20 cards. There are 14 different yellow cards and 6 different blue cards.
(a) How many different sets of 4 cards can be dealt?
(b) A player is dealt 4 cards. What is the probability that they are all blue?
(c) How many ways are there of getting 2 yellow cards and 2 blue cards?
(d) How many ways are there of getting at least 1 blue card?

8. A college offers 7 revision sessions for Mathematics and 5 revision sessions for Chemistry.
(a) Leila wants to attend 5 revision sessions. How many different choices can she make?
(b) Joel wants to attend 2 Mathematics sessions and 3 Chemistry sessions. In how many ways can he choose which sessions to attend?
(c) Frank randomly selects 5 sessions. Find the probability that at least one of them is Mathematics.

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1.3 CHOOSING FROM GROUPS WHEN ORDER DOES MATTER

WORKED EXAMPLE 1.3

Ten athletes compete in a race.

(a) In how many different ways can the first three places be filled?
(b) In how many different ways can the first three places be filled if Carl is one of them?

(a) The number of ways to select and place 3 athletes out of the 10 is ${}^{10}P_3 = 720$.

> The order of the placing in the race does matter, so use ${}^nP_r$.

(b) There are 3 options for Carl's position.

Once his position is fixed, there are ${}^9P_2 = 72$ ways of placing 2 other athletes out of the remaining 9.

Therefore, there are $3 \times 72 = 216$ ways.

> Deal with the constraint first: there are a limited number of options for Carl's position.

Practice questions 1.3

9. Rachel has nine cards, each with a different number from 1–9 written on them. She selects six cards at random, to form a six-digit number. How many different six-digit numbers are possible?

10. A bag contains 26 tiles, each with a different letter.
(a) Player 1 selects four tiles and places them in order. How many different 'words' can he get?
(b) Player 2 selects another four tiles, one at a time, and places them directly after the first player's tiles. How many different eight-letter 'words' can they form in this way?

11. (a) A car registration number consists of two different digits chosen from 1–9 followed by three different letters. How many possible registration numbers are there?
(b) How many possible registration numbers are there if the digits can be repeated (but the letters cannot)?

12. Ten athletes compete in a race. In how many different ways can the three medals be awarded if Sally wins either a gold or a silver?

13. Four letters are chosen from the word 'EQUATIONS' and arranged in order. How many of the possible arrangements contain at least one consonant?

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1.4 SOLVING EQUATIONS WITH BINOMIAL COEFFICIENTS AND FACTORIALS

WORKED EXAMPLE 1.4

Solve the equation $\binom{n}{3} = 220$.

$\binom{n}{3} = 220$

$\Leftrightarrow \frac{n(n-1)(n-2)}{6} = 220$

This is a cubic equation, which is not easy to solve by factorisation.

Put equations of this form straight into your calculator. Avoid doing any manipulations as this may give rise to mistakes.

From a table on the GDC, $n = 12$.

As n is a whole number, it is easier to use a table rather than a graph. We need to extend the table until we find the answer.

Check your answer! $\binom{12}{3} = 220$, so you know your answer is correct.

Practice questions 1.4

14. Solve the equation $\binom{n}{2} = 28$.

Remember that: $\binom{n}{1} = n$; $\binom{n}{2} = \frac{n(n-1)}{2}$; $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

15. Ahmed has some marbles. He lists all the possible ways of selecting three marbles, and finds that there are 35 possibilities. How many marbles does Ahmed have?

16. Solve the equation $(n + 1)! = 110(n - 1)!$

17. Solve the equation $n! + (n - 1)! = 24(n - 2)!$

The link between one factorial and the next can be shown algebraically as: $(n + 1)! = (n + 1) \times n!$

18. Solve the equation ${}^nP_4 = 12 \times {}^nP_3$.

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Mixed practice 1

1. Five students are to be selected from a class of 12 to go on a Mathematics trip.
   (a) How many possible selections are there?
   (b) What is the probability that Jonny and Lin are both selected?

2. A car registration number consists of two different letters followed by six digits chosen from 1–9 (the digits can be repeated). How many different registration numbers are possible?

3. Solve the equation $\binom{n}{2} = 210$.

4. A football team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 strikers. The coach needs to pick the team from a squad of 3 goalkeepers, 7 defenders, 9 midfielders and 3 strikers. In how many different ways can he pick the team?

5. How many different three-digit numbers can be formed from the digits 1–7:
   (a) if no digit can be repeated?
   (b) if repetitions are allowed?

6. Olivia has 5 green pens and 3 purple pens. She needs to select 4 pens to put in her bag.
   (a) How many possible selections include 2 green and 2 purple pens?
   (b) How many possible selections include at least 1 purple pen?

7. A bag contains six tiles with the letters A, E, U, D, G and Z. The tiles are selected at random and arranged to form a 'word'. What is the probability that this permutation starts and ends with a consonant?

8. Daniel and Theo each have n tiles with a different letter on each. Daniel picks three tiles and arranges them in a sequence (so ABC is a different sequence from ACB).
   (a) Write down an expression for the number of possible sequences Daniel can make.

   Daniel returns his tiles and then Theo selects three tiles without arranging them.

   (b) Given that Daniel can make 50 more sequences than Theo can make selections, find the value of n.

9. (a) Write down the number of possible permutations of the word 'COMPUTER'.
   (b) How many of the permutations have all the vowels together?
   (c) How many of the permutations end in TER?
   (d) How many of the permutations have the letters T, E, R next to each other?

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10. A class contains 10 boys and 12 girls, all with different names.

(a) Find an expression for the number of ways in which they could be arranged for a photo if:

(i) it is in one long line of 22 people

(ii) it is in two rows of 11 people.

(b) The teacher wants to form a committee from the class which contains Max, Lizzy, Jessie and two other students. In how many ways can this be done?

(c) Another group of students consists of 3 girls and 3 boys. They form a line for a photo. In how many ways can this be done if:

(i) all the boys are kept together?

(ii) all the boys are kept apart?

(d) Another class of n students have a race. There are 1320 different ways of awarding medals for the top three places (assuming that there are no ties). Find n.

Going for the top 1

1. How many permutations of the word 'SELECTION' have all of the vowels separated?

2. A four-digit number is formed from the digits 1, 2, 3, 4 and 5.

(a) How many such numbers can be formed if each digit can be used only once?

(b) How many such numbers can be formed if each digit can be used more than once?

(c) If each digit can be used only once, how many of the numbers are greater than 3000 and even?

3. How many distinct ways are there of arranging the letters AAABBBB?

4. (a) Show that the equation $\binom{n}{3} = \binom{n+1}{2} - 5$, can be written in the form $f(n) = 0$, where $f(x) = x^3 - 6x^2 - x + 30$.

(b) Show that $(x+2)$ is a factor of $f(x)$.

(c) Hence find all the solutions of the equation $\binom{n}{3} = \binom{n+1}{2} - 5$.

2 EXPONENTS AND LOGARITHMS

WHAT YOU NEED TO KNOW

- The rules of exponents:
  - $a^m \times a^n = a^{m+n}$
  - $\dfrac{a^m}{a^n} = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
  - $a^{-n} = \dfrac{1}{a^n}$
  - $a^n \times b^n = (ab)^n$
  - $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$
- The relationship between exponents and logarithms:
  - $a^x = b \Leftrightarrow x = \log_a b$ where a is called the base of the logarithm
  - $\log_a a^x = x$
  - $a^{\log_a x} = x$
- The rules of logarithms:
  - $\log_a x + \log_a y = \log_a xy$
  - $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$
  - $k \log_a x = \log_a x^k$
  - $\log_a \left(\dfrac{1}{x}\right) = -\log_a x$
  - $\log_a 1 = 0$
- The change of base rule: $\log_b a = \dfrac{\log_c a}{\log_c b}$
- There are two common abbreviations for logarithms to particular bases:
  - $\log_{10} x$ is often written as $\log x$
  - $\log_e x$ is often written as $\ln x$

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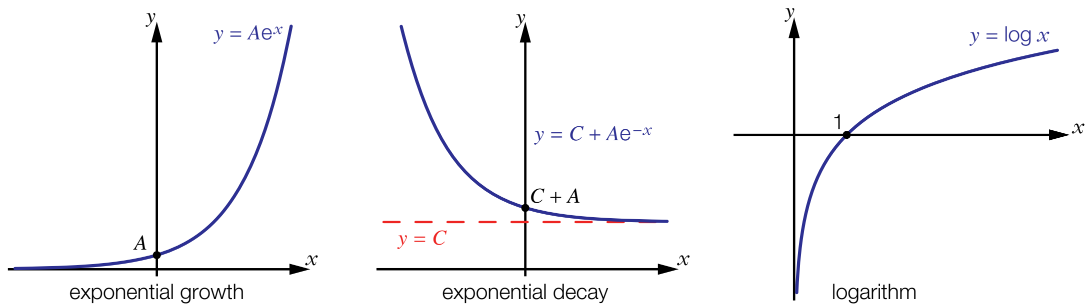
- The graphs of exponential and logarithmic functions:



EXAM TIPS AND COMMON ERRORS

- You must know what you *cannot* do with logarithms:
  - $\log(x + y)$ cannot be simplified; it is **not** $\log x + \log y$
  - $\log(e^x + e^y)$ cannot be simplified; it is **not** $x + y$
  - $(\log x)^2$ is **not** $2\log x$, whereas $\log x^2 = 2\log x$
  - $e^{2 + \log x} = e^2 e^{\log x} = e^2 x$ **not** $e^2 + x$