1 One-Dimensional, Steady-State Conduction

## 1.1 Conduction Heat Transfer

1.1.1 Introduction

Thermodynamics defines heat as a transfer of energy across the boundary of a system as a result of a temperature difference. According to this definition, heat by itself is an energy transfer process and it is therefore redundant to use the expression 'heat transfer'. Heat has no option but to transfer and the expression 'heat transfer' reinforces the incorrect concept that heat is a property of a system that can be 'transferred' to another system. This concept was originally proposed in the 1800's as the caloric theory (Keenan, 1958); heat was believed to be an invisible substance (having mass) that transferred from one system to another as a result of a temperature difference. Although the caloric theory has been disproved, it is still common to refer to 'heat transfer'.

Heat is the transfer of energy due to a temperature gradient. This transfer process can occur by two very different mechanisms, referred to as conduction and radiation. Conduction heat transfer occurs due to the interactions of molecular (or smaller) scale energy carriers within a material. Radiation heat transfer is energy transferred as electromagnetic waves. In a flowing fluid, conduction heat transfer occurs in the presence of energy transfer due to bulk motion (which is not a heat transfer) and this leads to a substantially more complex situation that is referred to as convection.

1.1.2 Thermal Conductivity

Conduction heat transfer occurs due to the interactions of micro-scale energy carriers within a material; the type of energy carriers depends upon the structure of the material. For example, in a gas or a liquid, the energy carriers are individual molecules whereas the energy carriers in a solid may be electrons or phonons (i.e., vibrations in the structure of the solid). The transfer of energy by conduction is fundamentally related to the interactions of these energy carriers; more energetic (i.e., higher temperature) energy carriers transfer energy to less energetic (i.e., lower temperature) ones, resulting in a net flow of energy from hot to cold (i.e., heat transfer). Regardless of the type of energy carriers involved, conduction heat transfer can be characterized by Fourier's law, provided that the length and time scales of the problem are large relative to the distance and time between energy carrier interactions. Fourier's law relates the heat flux in any direction to the temperature gradient in that direction. For example:

$$\dot{q}'' = -k \, \frac{\partial T}{\partial x} \qquad (1\text{-}1)$$

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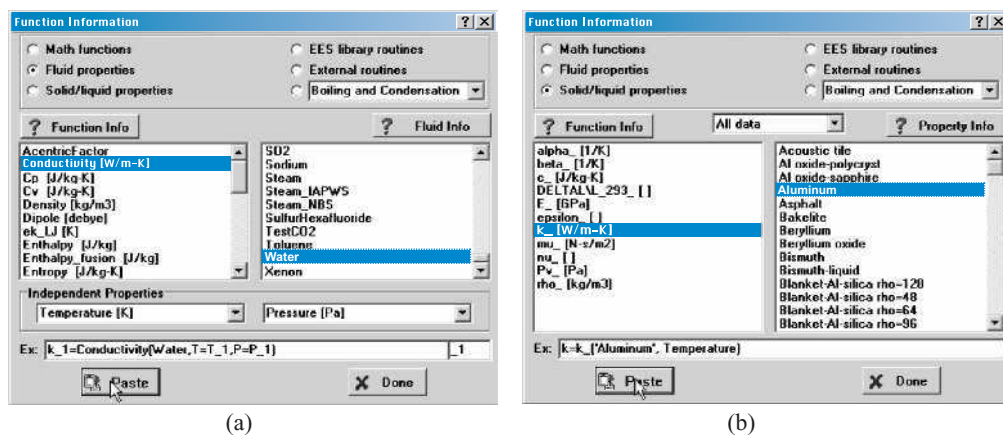
Figure 1-1: Conductivity functions in EES for (a) compressible substances and (b) incompressible substances.

where $\dot{q}''$ is the heat flux in the *x*-direction and *k* is the thermal conductivity of the material. Fourier's law actually provides the definition of thermal conductivity:

$$k = \frac{-\dot{q}''}{\frac{\partial T}{\partial x}} \tag{1-2}$$

Thermal conductivity is a material property that varies widely depending on the type of material and its state. Thermal conductivity has been extensively measured and values have been tabulated in various references (e.g., NIST (2005)). The thermal conductivity of many substances is available within the Engineering Equation Solver (EES) program. It is suggested that the reader go through the tutorial that is provided in Appendix A.1 in order to become familiar with EES. Appendix A.1 can be found on the web site associated with this book (www.cambridge.org/nellisandklein). To access the thermal conductivity functions in EES, select Function Info from the Options menu and select the Fluid Properties button; this action displays the properties that are available for compressible fluids. Navigate to Conductivity in the left hand window and select the fluid of interest in the right hand window (e.g., Water), as shown in Figure 1-1(a). Select Paste in order to place the call to the Conductivity function into the Equations Window. Select the Solid/liquid properties button in order to access the properties for incompressible fluids and solids, as shown in Figure 1-1(b).

The EES code below specifies the unit system to be used (SI) and then computes the conductivity of water, air, and aluminum (k_w, k_a, and k_{al}) at $T = 20°C$ and $p = 1.0$ atm,

```
$UnitSystem SI MASS RAD PA K J
$Tabstops 0.2 0.4 0.6 3.5 in

T=converttemp(C,K,20 [C])                "temperature"
P=1.0 [atm]*convert(atm,Pa)              "pressure"
k_w=Conductivity(Water,T=T,P=P)          "conductivity of water at T and P"
k_a=Conductivity(Air,T=T)                "conductivity of air at T and P"
k_al=k_('Aluminum', T)                   "conductivity of aluminum at T"
```

which leads to $k_w = 0.59$ W/m-K, $k_a = 0.025$ W/m-K, and $k_{al} = 236$ W/m-K. The conductivity of aluminum (an electrically conductive metal) is approximately 10,000x that
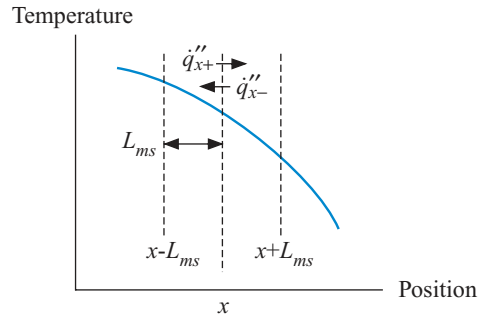
Figure 1-2: Energy flow through a plane at position x.

of air (a dilute gas, at these conditions), with water (a liquid) falling somewhere between these values.

It is possible to understand the thermal conductivity of various materials based on the underlying characteristics of their energy carriers, the microscopic physical entities that are responsible for conduction. For example, the kinetic theory of gases may be used to provide an estimate of the thermal conductivity of a gas and the thermal conductivity of electrically conductive metals can be understood based on a careful study of electron behavior.

Consider conduction through a material in which a temperature gradient has been established in the x-direction, as shown in Figure 1-2. We can evaluate (approximately) the net rate of energy transferred through a plane that is located at position x. The flux of energy carriers passing through the plane from left-to-right (i.e., in the positive x-direction) is proportional to the number density of the energy carriers (n_{ms}) and their mean velocity (v_{ms}). The energy carriers that are moving in the positive x-direction experienced their last interaction at approximately $x–L_{ms}$ (on average), where L_{ms} is the distance between energy carrier interactions. (Actually, the last interaction would not occur exactly at this position since the energy carriers are moving relative to each other and also in the y- and z-directions.) The energy associated with these left-to-right moving carriers is proportional to the temperature at position x-L_{ms} ($T_{x-L_{ms}}$). The energy per unit area passing through the plane from left-to-right ($\dot{q}''_{x+}$) is given approximately by:

$$\dot{q}''_{x+} \approx \underbrace{n_{ms}\, v_{ms}}_{\substack{\text{\#carriers} \\ \text{area-time}}}\, \underbrace{c_{ms}\, T_{x-L_{ms}}}_{\substack{\text{energy} \\ \text{carrier}}} \tag{1-3}$$

where c_{ms} is the ratio of the energy of the carrier to its temperature. Similarly, the energy per unit area carried through the plane in the negative x-direction by the energy carriers that are moving from right-to-left ($\dot{q}''_{x-}$) is given approximately by:

$$\dot{q}''_{x-} \approx n_{ms}\, v_{ms}\, c_{ms}\, T_{x+L_{ms}} \tag{1-4}$$

The net conduction heat flux passing through the plane ($\dot{q}''$) is the difference between $\dot{q}''_{x+}$ and $\dot{q}''_{x-}$,

$$\dot{q}'' \approx n_{ms}\, v_{ms}\, c_{ms}\, (T_{x-L_{ms}} - T_{x+L_{ms}}) \tag{1-5}$$

which can be rearranged to yield:

$$\dot{q}'' \approx -n_{ms}\, v_{ms}\, c_{ms}\, L_{ms}\, \frac{(T_{x+L_{ms}} - T_{x-L_{ms}})}{L_{ms}} \tag{1-6}$$

Recall from calculus that the definition of the temperature gradient is:

$$\frac{\partial T}{\partial x} = \lim_{dx \to 0} \frac{T_{x+dx} - T_{x-dx}}{2\, dx} \tag{1-7}$$
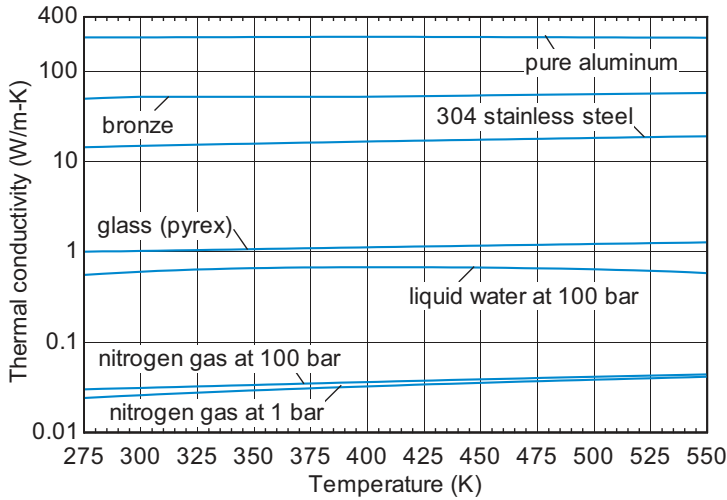
Figure 1-3: Thermal conductivity of various materials as a function of temperature.

In the limit that the length between energy carrier interactions (L_{ms}) is much less than the length scale that characterizes the problem (L_{char}):

$$\frac{(T_{x+L_{ms}} - T_{x-L_{ms}})}{2\,L_{ms}} \approx \lim_{dx \to 0} \frac{T_{x+dx} - T_{x-dx}}{2\,dx} = \frac{\partial T}{\partial x} \tag{1-8}$$

Equation (1-8) can be substituted into Eq. (1-6) to yield:

$$\dot{q}'' \approx -2 \underbrace{n_{ms}\,v_{ms}\,c_{ms}\,L_{ms}}_{\propto k}\ \frac{\partial T}{\partial x} \tag{1-9}$$

The ratio of the length between energy carrier interactions to the length scale that characterizes the problem is referred to as the Knudsen number. The Knudsen number (Kn) should be calculated in order to ensure that continuum concepts (like Fourier's law) are applicable:

$$Kn = \frac{L_{ms}}{L_{char}} \tag{1-10}$$

If the Knudsen number is not small then continuum theory breaks down. This limit may be reached in micro- and nano-scale systems where L_{char} becomes small as well as in problems involving rarefied gas where L_{ms} becomes large. Specialized theory for heat transfer is required in these limits and the interested reader is referred to books such as Tien et al. (1998), Chen (2005), and Cercignani (2000).

Comparing Eq. (1-9) with Fourier's law, Eq. (1-1), shows that the thermal conductivity is proportional to the product of the number of energy carriers per unit volume, their average velocity, the mean distance between their interactions, and the ratio of the amount of energy carried by each energy carrier to its temperature:

$$k \propto n_{ms}\,v_{ms}\,c_{ms}\,L_{ms} \tag{1-11}$$

The scaling relation expressed by Eq. (1-11) is informative. Figure 1-3 illustrates the thermal conductivity of several common materials as a function of temperature.

Notice that metals have the largest thermal conductivity, followed by other solids and liquids, while gases have the lowest conductivity. Gases are diffuse and thus the number density of the energy carriers (gas molecules) is substantially less than for other

forms of matter. Pure metals have the highest thermal conductivity because energy is carried primarily by electrons which are numerous and fast moving. The thermal conductivity and electrical resistivity of pure metals are related (by the Wiedemann-Franz law) because both electricity and thermal energy are transported by the same mechanism, electron flow. Alloys have lower thermal conductivity because the electron motion is substantially impeded by the impurities within the structure of the material; this effect is analogous to reducing the parameter L_{ms} in Eq. (1-11). In non-metals, the energy is carried by phonons (or lattice vibrations), while in liquids the energy is carried by molecules.

Thermal Conductivity of a Gas

This extended section of the book can be found on the website (www.cambridge.org/ nellisandklein) and discusses the application of Eq. (1-11) to the particular case of an ideal gas where the energy carriers are gas molecules.

1.2 Steady-State 1-D Conduction without Generation

1.2.1 Introduction

Chapters 1 through 3 examine conduction problems using a variety of conceptual, analytical, and numerical techniques. We will begin with simple problems and move eventually to complex problems, starting with truly one-dimensional (1-D), steady-state problems and working finally to two-dimensional and transient problems. Throughout this book, problems will be solved both analytically and numerically. The development of an analytical or a numerical solution is accomplished using essentially the same steps regardless of the complexity of the problem; therefore, each class of problem will be solved in a uniform and rigorous fashion. The use of computer software tools facilitates the development of both analytical and numerical solutions; therefore, these tools are introduced and used side-by-side with the theory.

1.2.2 The Plane Wall

In general, the temperature in a material will be a function of position (x, y, and z, in Cartesian coordinates) and time (t). The definition of steady-state is that the temperature is unchanging with time. There are certain idealized problems in which the temperature varies in only one direction (e.g., the x-direction). These are one-dimensional (1-D), steady-state problems. The classic example is a plane wall (i.e., a wall with a constant cross-sectional area, A_c, in the x-direction) that is insulated around its edges. In order for the temperature distribution to be 1-D, each face of the wall must be subjected to a uniform boundary condition. For example, Figure 1-4 illustrates a plane wall in which the left face ($x = 0$) is maintained at T_H while the right face ($x = L$) is held at T_C.
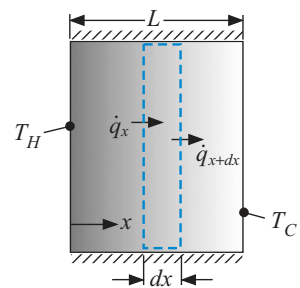
Figure 1-4: A plane wall with fixed temperature boundary conditions.

The first step toward developing an analytical solution for this, or any, problem involves the definition of a differential control volume. The control volume must encompass material at a uniform temperature; therefore, in this case it must be differentially small in the x-direction (i.e., it has width dx, see Figure 1-4) but can extend across the entire cross-sectional area of the wall as there are no temperature gradients in the y- or z-directions. Next, the energy transfers across the control surfaces must be defined as well as any thermal energy generation or storage terms. For the steady-state, 1-D case considered here, there are only two energy transfers, corresponding to the rate of conduction heat transfer into the left side (i.e., at position x, $\dot{q}_x$) and out of the right side (i.e., at position $x + dx$, $\dot{q}_{x+dx}$) of the control volume. A steady-state energy balance for the differential control volume is therefore:

$$\dot{q}_x = \dot{q}_{x+dx} \tag{1-19}$$

A Taylor series expansion of the term at $x + dx$ leads to:

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}}{dx}\,dx + \frac{d^2\dot{q}}{dx^2}\,\frac{dx^2}{2!} + \frac{d^3\dot{q}}{dx^3}\,\frac{dx^3}{3!} + \cdots \tag{1-20}$$

The analytical solution proceeds by taking the limit as dx goes to zero so that the higher order terms in Eq. (1-20) can be neglected:

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}}{dx}\,dx \tag{1-21}$$

Substituting Eq. (1-21) into Eq. (1-19) leads to:

$$\dot{q}_x = \dot{q}_x + \frac{d\dot{q}}{dx}\,dx \tag{1-22}$$

or

$$\frac{d\dot{q}}{dx} = 0 \tag{1-23}$$

Equation (1-23) is typical of the initial result that is obtained by considering a differential energy balance: a differential equation that is expressed in terms of energy rather than temperature. This form of the differential equation should be checked against your intuition. Equation (1-23) indicates that the rate of conduction heat transfer is not a function of x. For the problem in Figure 1-4, there are no sources or sinks of energy and no energy storage within the wall; therefore, there is no reason for the rate of heat transfer to vary with position.

The final step in the derivation of the governing equation is to substitute appropriate rate equations that relate energy transfer rates to temperatures. The result of this substitution will be a differential equation expressed in terms of temperature. The rate equation for conduction is Fourier's law:

$$\dot{q} = -k\,A_c\,\frac{\partial T}{\partial x} \tag{1-24}$$

For our problem, the temperature is only a function of position, x, and therefore the partial differential in Eq. (1-24) can be replaced with an ordinary differential:

$$\dot{q} = -k\,A_c\,\frac{dT}{dx} \tag{1-25}$$

Substituting Eq. (1-25) into Eq. (1-23) leads to:

$$\frac{d}{dx}\left[-k\,A_c\,\frac{dT}{dx}\right] = 0 \tag{1-26}$$

If the thermal conductivity is constant then Eq. (1-26) may be simplified to:

$$\frac{d^2 T}{dx^2} = 0 \qquad (1\text{-}27)$$

The derivation of Eq. (1-27) is trivial and yet the steps are common to the derivation of the governing equation for more complex problems. These steps include: (1) the definition of an appropriate control volume, (2) the development of an energy balance, (3) the expansion of terms, and (4) the substitution of rate equations.

In order to completely specify a problem, it is necessary to provide boundary conditions. Boundary conditions are information about the solution at the extents of the computational domain (i.e., the limits of the range of position and/or time over which your solution is valid). A second order differential equation requires two boundary conditions. For the problem shown in Figure 1-4, the boundary conditions are:

$$T_{x=0} = T_H \qquad (1\text{-}28)$$

$$T_{x=L} = T_C \qquad (1\text{-}29)$$

Equations (1-27) through (1-29) represent a well-posed mathematical problem: a second order differential equation with boundary conditions. Equation (1-27) is very simple and can be solved by separation and direct integration:

$$d\left[\frac{dT}{dx}\right] = 0 \qquad (1\text{-}30)$$

Equation (1-30) is integrated according to:

$$\int d\left[\frac{dT}{dx}\right] = \int 0 \qquad (1\text{-}31)$$

Because Eq. (1-31) is an indefinite integral (i.e., there are no limits on the integrals), an undetermined constant (C_1) results from the integration:

$$\frac{dT}{dx} = C_1 \qquad (1\text{-}32)$$

Equation (1-32) is separated and integrated again:

$$\int dT = \int C_1 \, dx \qquad (1\text{-}33)$$

to yield

$$T = C_1 x + C_2 \qquad (1\text{-}34)$$

Equation (1-34) shows that the temperature distribution must be linear; any linear function (i.e., any values of the constants C_1 and C_2) will satisfy the differential equation, Eq. (1-27). The constants of integration are obtained by forcing Eq. (1-34) to also satisfy the two boundary conditions, Eqs. (1-28) and (1-29):

$$T_H = C_1 \, 0 + C_2 \qquad (1\text{-}35)$$

$$T_C = C_1 \, L + C_2 \qquad (1\text{-}36)$$

Equations (1-35) and (1-36) are solved for C_1 and C_2 and substituted into Eq. (1-34) to provide the solution:

$$T = \frac{(T_C - T_H)}{L} x + T_H \qquad (1\text{-}37)$$

The heat transfer at any location within the wall is obtained by substituting the temperature distribution, Eq. (1-37), into Fourier's law, Eq. (1-25):

$$\dot{q} = -k\,A_c\,\frac{dT}{dx} = \frac{k\,A_c}{L}\,(T_H - T_C) \tag{1-38}$$

Equation (1-38) shows that the heat transfer does not change with the position within the wall; this behavior is consistent with Eq. (1-23).

The development of analytical solutions is facilitated using a symbolic software package such as Maple. It is suggested that the reader stop and go through the tutorial provided in Appendix A.2 which can be found on the web site associated with the book (www.cambridge.org/nellisandklein) in order to become familiar with Maple. Note that the Maple Command Applet that is discussed in Appendix A.2 is available on the internet and can be used even if you do not have access to the Maple software. The mathematical solution to the 1-D, steady-state conduction problem associated with a plane wall is easy enough that there is no reason to use Maple. However, it is worthwhile to use the problem in order to illustrate some of the basic steps associated with using Maple in anticipation of more difficult problems. Start a new problem in Maple (select New from the File menu). Enter the governing differential equation, Eq. (1-27), and assign it to the function ODE; note that the second derivative of T with respect to x is obtained by applying the diff command twice.

```
> restart;
> ODE:=diff(diff(T(x),x),x)=0;
```
$$ODE := \frac{d^2}{dx^2}T(x) = 0$$

The solution to the ordinary differential equation is obtained using the dsolve command and assigned to the function Ts.

```
> Ts:=dsolve(ODE);
```
$$Ts := \mathrm{T}(x) = \_C1\,x + \_C2$$

The solution identified by Maple is consistent with Eq. (1-34), except that Maple uses the variables _C1 and _C2 rather than C_1 and C_2 to represent the constants of integration. The two boundary conditions, Eqs. (1-35) and (1-36), are obtained symbolically using the eval command to evaluate the solution at a particular position and assigned to the functions BC1 and BC2.

```
> BC1:=eval(Ts,x=0)=T_H;
```
$$BC1 := (\mathrm{T}(0) = \_C2) = T\_H$$
```
> BC2:=eval(Ts,x=L)=T_C;
```
$$BC2 := (T(L) = \_C1\,L + \_C2) = T\_C$$

The result of the eval command is almost, but not quite what is needed to solve for the constants. The expressions include the extraneous statements T(0) and T(L); use the rhs function in order to return just the expression on the right hand side.

```
> BC1:=rhs(eval(Ts,x=0))=T_H;
```
$$BC1 := \_C2 = T\_H$$
```
> BC2:=rhs(eval(Ts,x=L))=T_C;
```
$$BC2 := \_C1L + \_C2 = T\_C$$

The constants are explicitly determined using the solve command. Note that the solve command requires two arguments; the first is the equation or, in this case, set of equations to be solved (the boundary conditions, BC1 and BC2) and the second is the variable or set of variables to solve for (the constants _C1 and _C2).

```
> constants:=solve({BC1,BC2},{_C1,_C2});
```
$$constants := \{\_C2 = T\_H, \_C1 = -\frac{T\_H - T\_C}{L}\}$$

The constants are substituted into the general solution using the subs command. The subs command requires two arguments; the first is the set of definitions to be substituted and the second is the set of equations to substitute them into.

```
> Ts:=subs(constants,Ts);
```
$$Ts := \mathrm{T}(x) = -\frac{(T\_H - T\_C)x}{L} + T\_H$$

This result is the same as Eq. (1-37).

1.2.3 The Resistance Concept

Equation (1-38) is the solution for the rate of heat transfer through a plane wall. The equation suggests that, under some limiting conditions, conduction of heat through a solid can be thought of as a flow that is driven by a temperature difference and resisted by a thermal resistance, in the same way that electrical current is driven by a voltage difference and resisted by an electrical resistance. Inspection of Eq. (1-38) suggests that the thermal resistance to conduction through a plane wall (R_{pw}) is given by:

$$R_{pw} = \frac{L}{k A_c} \qquad (1\text{-}39)$$

allowing Eq. (1-38) to be rewritten:

$$\dot{q} = \frac{(T_H - T_C)}{R_{pw}} \qquad (1\text{-}40)$$

The concept of a thermal resistance is broadly useful and we will often return to this idea of a thermal resistance in order to help develop a conceptual understanding of various heat transfer processes. The usefulness of Eqs. (1-39) and (1-40) go beyond the simple situation illustrated in Figure 1-4. It is possible to approximately understand conduction heat transfer in most any situation provided that you can identify the distance that heat must be conducted and the cross-sectional area through which the conduction occurs.
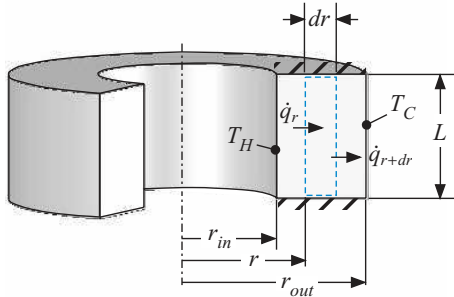
Figure 1-5: A cylinder with fixed temperature boundary conditions.

Resistance equations provide a method for succinctly summarizing a particular solution and we will derive resistance solutions for a variety of physical situations. By cataloging these resistance equations, it is possible to quickly use the solution in the context of a particular problem without having to go through all of the steps that were required in the original derivation. For example, if we are confronted with a problem involving steady-state heat transfer through a plane wall then it is not necessary to rederive Eqs. (1-19) through (1-38); instead, Eqs. (1-39) and (1-40) conveniently represent all of this underlying math.

1.2.4 Resistance to Radial Conduction through a Cylinder

Figure 1-5 illustrates steady-state, radial conduction through an infinitely long cylinder (or one with insulated ends) without thermal energy generation. The analytical solution to this problem is derived using the steps described in Section 1.2.2. The differential energy balance (see Figure 1-5) leads to:

$$\dot{q}_r = \dot{q}_{r+dr} \tag{1-41}$$

The $r + dr$ term in Eq. (1-41) is expanded and Fourier's law is substituted in order to reach:

$$\frac{d}{dr}\left[-k\,A_c\,\frac{dT}{dr}\right] = 0 \tag{1-42}$$

The difference between the plane wall geometry considered in Section 1.2.2 and the cylindrical geometry considered here is that the cross-sectional area for heat transfer, A_c in Eq. (1-42), is not constant but rather varies with radius:

$$\frac{d}{dr}\left[-k\,\underbrace{2\,\pi\,r\,L}_{A_c}\,\frac{dT}{dr}\right] = 0 \tag{1-43}$$

where L is the length of the cylinder. Assuming that the thermal conductivity is constant, Eq. (1-43) is simplified to:

$$\frac{d}{dr}\left[r\,\frac{dT}{dr}\right] = 0 \tag{1-44}$$

and integrated twice according to the following steps:

$$\int d\left[r\,\frac{dT}{dr}\right] = \int 0 \tag{1-45}$$

$$r\,\frac{dT}{dr} = C_1 \tag{1-46}$$