

**PART ONE**  
FLUID DYNAMICS AND WAVES

# 1

## Elements of fluid dynamics

It is convenient to start with a brief summary of fluid dynamics fundamentals in order to establish the mathematical notation and the physical concepts that will be used throughout this book. We will first look at the kinematics of fluid flow, especially at how to capture the evolution of material elements such as material points or lines.

This is followed by a description of perfect fluid dynamics, which is the natural point of departure for the study of flows at very high Reynolds numbers in the atmosphere and the ocean. In these flows the direct influence of viscous forces is confined to boundary layers and to sparse pockets of three-dimensional turbulence within the fluid.

The culmination of perfect fluid dynamics is Kelvin's circulation theorem and the various links of this theorem to vorticity dynamics. Indeed, as we go on it will become increasingly clear that the circulation theorem is also the key result in wave-mean interaction theory.

### 1.1 Flow kinematics

In continuum fluid mechanics the molecular structure of the fluid is ignored and the description of the physical state of the fluid is accomplished by specifying a finite number of flow fields as functions of position  $\boldsymbol{x}$  and time  $t$ , say. How many fields are needed depends on the complexity of the fluid under consideration, but all fluid flows require a working mass and momentum budget, which leads to the definitions of the density and velocity fields.

#### 1.1.1 Mass, momentum and velocity

The *mass density* per unit volume  $\rho(\boldsymbol{x}, t) \geq 0$  is defined such that

$$M = \int_{\mathcal{D}} \rho \, dV \tag{1.1}$$

is the fluid mass contained in any region  $\mathcal{D}$ , including the limiting case as  $\mathcal{D}$  shrinks to an infinitesimal region. The essential usage of  $\rho$  as an integrand suggests that  $\rho(\mathbf{x}, t)$  need not be very smooth; indeed, it certainly makes sense to allow for *discontinuity surfaces*, say at the interface between two fluids such as the interface between water and air, where the density jumps by a factor of a thousand, or at internal jumps such as compressible shocks in gas dynamics. Some fluids are mixtures that consist of several fluid species sharing the same volume and then each species has its own mass density. Examples occur in plasma physics, in superfluids, or even in the more familiar situation of rain drops or ice particles embedded in moist air. However, we will not consider mixtures in this book, so there will only be one fluid mass density.

The *momentum density* defines the fluid velocity vector  $\mathbf{u}(\mathbf{x}, t)$  such that

$$\int_{\mathcal{D}} \rho \mathbf{u} \, dV \quad (1.2)$$

is the fluid momentum contained in any region  $\mathcal{D}$ , again including the infinitesimal limit. In other words, the fluid velocity at a point  $\mathbf{x}$  is defined as the ratio between momentum and mass as  $\mathcal{D}$  shrinks to zero volume around  $\mathbf{x}$ . This is a basic statement of physics.

Another basic statement of physics is that mass is conserved, which means that  $M$  in (1.1) should be constant if the region  $\mathcal{D}$  moves with the fluid. More precisely, if we consider a region  $\mathcal{D}$  whose points move with velocity  $\mathbf{v}$ , say, then it follows from calculus that

$$\frac{dM}{dt} = \int_{\mathcal{D}} \frac{\partial \rho}{\partial t} \, dV + \oint_{\partial \mathcal{D}} \rho \mathbf{v} \cdot \mathbf{dA} = \int_{\mathcal{D}} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \, dV \quad (1.3)$$

where  $\mathbf{dA} = \mathbf{n} \, dA$  is the outward-pointing area element at the boundary  $\partial \mathcal{D}$  and the second form has been obtained by using the divergence theorem. Now, the physical statement of *mass conservation* implies that (1.3) is zero if  $\mathbf{v} = \mathbf{u}$ , in which case  $\mathcal{D}$  is called a material volume and (1.3) is called the *continuity equation* in integral form. For this to be true for any choice of  $\mathcal{D}$  it must be the case that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.4)$$

This is the *continuity equation* in differential form. It is worth noting the double role of  $\rho \mathbf{u}$ , which is both the momentum vector density and also the mass flux vector.

### 1.1.2 Material trajectories and derivatives

The integral curves of  $\mathbf{u}$  are called *material trajectories* and they are given by the vector-valued functions  $\mathbf{X}(t)$  that solve the initial-value problem

$$t \in [0, T]: \quad \frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}(t), t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{X}_0 \quad (1.5)$$

with some initial position  $\mathbf{X}_0$ . Thus,  $\mathbf{X}(t)$  is the trajectory during the time interval  $[0, T]$  of the *fluid particle*<sup>1</sup> that was at position  $\mathbf{X}_0$  at time  $t = 0$ . It is a working assumption in fluid dynamics that (1.5) is well posed for some finite time of interest  $T > 0$ , which implies that during this time interval material trajectories are unique and do not cross. Thus a fluid particle is uniquely identified, or labelled, by its initial position  $\mathbf{X}_0$ . Now, using  $\mathbf{X}(t)$  we can evaluate any field  $\phi(\mathbf{x}, t)$  along the trajectory  $\mathbf{x} = \mathbf{X}(t)$  to obtain the time evolution of  $\phi$  as observed by following the particle. Specifically, the chain rule yields

$$\frac{d}{dt}\phi(\mathbf{X}(t), t) = \left( \frac{\partial}{\partial t} + \mathbf{u}(\mathbf{X}(t), t) \cdot \nabla \right) \phi(\mathbf{X}(t), t) \quad (1.6)$$

for the rate of change of  $\phi$  along the material trajectory. This shows how the velocity field  $\mathbf{u}$  defines a directional derivative following the fluid motion, which is called the *material derivative*. Obviously, the material derivative can also be expressed without reference to  $\mathbf{X}(t)$  by

$$\frac{D\phi}{Dt} \equiv \frac{\partial\phi}{\partial t} + (\mathbf{u} \cdot \nabla)\phi. \quad (1.7)$$

This gives the rate of change per unit time of  $\phi$  as observed by following the fluid particle that is occupying the position  $\mathbf{x}$  at time  $t$ . This is the most important mathematical definition in fluid dynamics.

For smooth flows the continuity equation (1.4) can be re-written in the equivalent form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad (1.8)$$

which shows explicitly how the density following a particle evolves according to the divergence of the velocity field. In the special case of *incompressible*

<sup>1</sup> The concept of a ‘fluid particle’ here must not be confused with a fluid molecule or any other form of atomic particle, which is necessarily part of the molecular structure that is ignored in a continuum theory. Instead, the fluid particle here is a small region  $\mathcal{D}$  surrounding a point  $\mathbf{x}$  such that the variations of  $\mathbf{u}$  across  $\mathcal{D}$  are negligible. Although  $\mathcal{D}$  is small in this sense it still contains very many atomic particles. This is a well-defined concept in smooth flows although some care is needed in the presence of discontinuity surfaces.

flow  $D\rho/Dt = 0$  and we have the pair

$$\frac{D\rho}{Dt} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0. \quad (1.9)$$

Incompressibility does not imply that  $\rho$  is spatially uniform, only that  $\rho$  is constant along material trajectories. This is an example of *material invariance*, which takes the form (1.9a) for scalar fields.

### 1.1.3 Lagrangian and Eulerian variables

In fluid kinematics there is a distinction between *Eulerian variables* that are defined as functions of fixed position  $\mathbf{x}$  and *Lagrangian variables* that are defined relative to fixed fluid particles labelled by their initial positions  $\mathbf{X}_0$ , say. For example, we can think of the mass density as a Eulerian variable described by  $\rho(\mathbf{x}, t)$  or as a Lagrangian variable described by a function of the initial particle positions  $\rho(\mathbf{X}_0, t)$ . Clearly, this choice affects the form of the governing equations, for instance the material derivative takes the alternative forms

$$\frac{D\rho}{Dt} = \left( \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \right) \rho(\mathbf{x}, t) = \frac{\partial}{\partial t} \rho(\mathbf{X}_0, t). \quad (1.10)$$

Both descriptions are mathematically equivalent, but in practice (especially in numerical computations) the use of Eulerian variables dominates. This is because realistic fluid flows involve large, chaotic particle trajectories that would render a mathematical description or numerical computation based on exactly resolved trajectories infeasible after a very short time. This is obvious from everyday experiences such as stirring milk in coffee, for example. Nevertheless, Lagrangian variables have their conceptual merits because they greatly simplify the description of material invariants. In addition, some dynamical laws such as Kelvin's circulation theorem can only be exploited fully using Lagrangian concepts.

We will take the view that fluid dynamics is naturally a hybrid theory in which both Eulerian and Lagrangian variables play a useful role in understanding and computation. This will also be our approach to wave–mean interaction theory.

### 1.1.4 Evolution of material elements

The material derivative (1.7) extends naturally to infinitesimal material elements such as points, lines, areas and volumes. The material derivative of

a point is

$$\frac{D\mathbf{x}}{Dt} = 0 + (\mathbf{u} \cdot \nabla)\mathbf{x} = \mathbf{u}. \quad (1.11)$$

Thus the material rate of change of position is given by the velocity. Two neighbouring material points  $\mathbf{x}$  and  $\mathbf{x} + d\mathbf{x}$  move with the respective velocities  $\mathbf{u}$  and  $\mathbf{u} + d\mathbf{u}$  and therefore a material line element  $d\mathbf{x}$  evolves according to

$$\frac{D}{Dt}(d\mathbf{x}) = d\mathbf{u} = (d\mathbf{x} \cdot \nabla)\mathbf{u}. \quad (1.12)$$

The evolution of the volume content of a material region follows from (1.3) with  $\rho = 1$  and  $\mathbf{v} = \mathbf{u}$  as

$$\frac{d}{dt} \int_{\mathcal{D}} dV = \int_{\mathcal{D}} \nabla \cdot \mathbf{u} dV \quad \Rightarrow \quad \frac{D}{Dt}(dV) = \nabla \cdot \mathbf{u} dV. \quad (1.13)$$

This shows that positive divergence corresponds to volume expansion. Combining (1.13) and (1.8) we obtain the continuity equation in the form

$$\frac{D}{Dt}(\rho dV) = 0, \quad (1.14)$$

i.e., the mass of a fluid particle is constant. This yields a useful general formula for the rate of change of an integral over a material volume  $\mathcal{D}$ , namely

$$\frac{d}{dt} \int_{\mathcal{D}} \rho \phi dV = \int_{\mathcal{D}} \frac{D}{Dt}(\rho \phi dV) = \int_{\mathcal{D}} \rho \frac{D\phi}{Dt} dV \quad (1.15)$$

for any function  $\phi$ . For example, this formula applies to the centre of mass of a material volume:

$$\frac{d}{dt} \left( \frac{1}{M} \int_{\mathcal{D}} \rho \mathbf{x} dV \right) = \frac{1}{M} \int_{\mathcal{D}} \rho \mathbf{u} dV. \quad (1.16)$$

Thus the centre of mass moves with the total momentum.

Finally, the evolution of a material area element  $d\mathbf{A}$  follows most easily from  $dV = d\mathbf{x} \cdot d\mathbf{A}$  and therefore

$$\frac{D}{Dt}(d\mathbf{x} \cdot d\mathbf{A}) = \frac{D d\mathbf{x}}{Dt} \cdot d\mathbf{A} + d\mathbf{x} \cdot \frac{D d\mathbf{A}}{Dt} = \nabla \cdot \mathbf{u} dV. \quad (1.17)$$

Substituting from (1.12) and demanding that the remaining equality holds for arbitrary  $d\mathbf{x}$  then implies

$$\frac{D}{Dt}(d\mathbf{A}) = (\nabla \cdot \mathbf{u}) d\mathbf{A} - (\nabla \mathbf{u}) \cdot d\mathbf{A}, \quad (1.18)$$

where in the last term  $d\mathbf{A}$  contracts with  $\mathbf{u}$  and not with  $\nabla$ . For example, in the case of isotropic expansion with  $\mathbf{u} = \mathbf{x}$  the right-hand side is  $(n-1)d\mathbf{A}$ ,

where  $n$  is the number of spatial dimensions. This situation is familiar from inflating a balloon. In non-isotropic flow fields the last term captures the tilting of  $d\mathbf{A}$  by the strain field.

For completeness, we note that the gradient of a material invariant  $\phi$  evolves according to

$$\frac{D\phi}{Dt} = 0 \quad \Rightarrow \quad \frac{D}{Dt}(\nabla\phi) = -(\nabla\mathbf{u}) \cdot \nabla\phi, \quad (1.19)$$

where again  $\nabla\phi$  contracts with  $\mathbf{u}$  and not with  $\nabla$ . For incompressible flows this is identical to the equation governing material area elements in (1.18). Experience shows that it is easy to get the signs wrong between (1.12), (1.18) and (1.19).

## 1.2 Perfect fluid dynamics

The equation of motion in a fluid follows from Newton's law, which demands that the time rate of change of momentum of a material body  $\mathcal{D}$ , say, is equal to the net force applied to the body. In a *perfect fluid* there are no viscous forces and this leads to Euler's equation. To close the set of equations we require constitutive relations, which for an ideal gas brings in the entropy. In perfect fluid flow there is no diabatic heating and therefore specific entropy is a material invariant. This closes the set of equations for perfect fluid flow.

### 1.2.1 Euler's equation

In a perfect fluid the only forces acting on the fluid are due to a potential per unit mass  $\Phi$  and a pressure  $p$ . By definition, the net force on  $\mathcal{D}$  is then

$$\int_{\mathcal{D}} -\rho\nabla\Phi dV - \oint_{\partial\mathcal{D}} p d\mathbf{A}. \quad (1.20)$$

For example, in the case of the standard gravitational potential  $\Phi = gz$  with altitude  $z$  and constant gravity  $g$  the net potential force is downward and equals the weight of the fluid in  $\mathcal{D}$ . There is a qualitative difference between these two kinds of forces: the potential force acts throughout the volume of  $\mathcal{D}$  whilst the pressure force acts only on its boundary  $\partial\mathcal{D}$ . Newton's law in integral form is

$$\frac{d}{dt} \int_{\mathcal{D}} \rho\mathbf{u} dV = \int_{\mathcal{D}} (-\rho\nabla\Phi - \nabla p) dV \quad (1.21)$$

and its differential form is *Euler's equation* for a perfect fluid:

$$\frac{D\mathbf{u}}{Dt} + \frac{\nabla p}{\rho} = -\nabla\Phi. \quad (1.22)$$

The division by  $\rho$  is convenient for vorticity dynamics. The appropriate boundary conditions for  $\mathbf{u}$  at a fixed wall are that the normal component of  $\mathbf{u}$  vanishes at the boundary. However, the tangential velocity components are not constrained in a perfect fluid.

### 1.2.2 Constitutive relations

We consider  $\Phi$  as given and then in  $n$  spatial dimensions the continuity and Euler equations provide  $1 + n$  equations for the  $2 + n$  fields  $(\rho, p, \mathbf{u})$ . Thus we need a relation that links  $\rho$  and  $p$ , which is usually derived from thermodynamics under the assumption of local thermodynamic equilibrium. However, in the case of incompressible flow this additional equation is simply  $\nabla \cdot \mathbf{u} = 0$ . In fact, this case is special because taking the divergence of (1.22) then leads to an elliptic equation for the pressure. This is a diagnostic equation, i.e., an equation without a time derivative, which makes clear that for incompressible flow the pressure is not an independent field, but can be computed, albeit non-locally, from the other flow fields.

Another important class of fluid models is that of a *barotropic fluid*, in which there is a non-degenerate functional relationship between density and pressure. This simple fluid model allows for compressible effects, and we will use it frequently in this book. The assumed functional relationship implies that the pressure force is irrotational in a barotropic fluid, i.e.,

$$\nabla \times \left( \frac{\nabla p}{\rho} \right) = \nabla \left( \frac{1}{\rho} \right) \times \nabla p = 0. \quad (1.23)$$

For fluids such as an *ideal gas* there is a functional relationship between density, pressure, and one more state variable  $s$ , which is the entropy density per unit mass. In a perfect fluid there is no diabatic heating due to irreversible processes or radiation effects and therefore entropy is materially invariant in smooth flow regions:

$$\frac{Ds}{Dt} = 0. \quad (1.24)$$

However, the entropy increases if fluid particles pass through a shock, as is well known in gas dynamics. The barotropic fluid can be viewed as a special case of an ideal gas in which  $s$  is spatially uniform; this is also called a *homentropic fluid*. So, in perfect smooth flow the continuity and Euler equations together with (1.24) and the functional relationship between  $(\rho, p, s)$  form a complete system for  $3 + n$  flow fields. This will be sufficient for the purposes of this book, but we note that more complex fluid models may require further thermodynamics fields, such as a humidity variable for moist



air, or they may include deviations from the thermodynamic relationships derived under the assumption of local thermodynamic equilibrium.

Finally, even for an ideal gas it is often useful to define some further thermodynamic flow fields, such as the temperature  $T$ , which are then linked to  $(\rho, p)$  via additional relations such as the ideal gas law and so on. For us it proves convenient to introduce the *internal energy* per unit mass, which is a function  $\epsilon(\rho, s)$  defined by the first law of thermodynamics:

$$d\epsilon = T ds - pd\left(\frac{1}{\rho}\right) = T ds + \frac{p}{\rho^2} d\rho \quad \Leftrightarrow \quad \frac{\partial\epsilon}{\partial\rho} = \frac{p}{\rho^2} \quad \text{and} \quad \frac{\partial\epsilon}{\partial s} = T. \quad (1.25)$$

This internal energy quantifies the change of elastic energy due to compression or dilation of fluid particles, which will be needed to formulate the energy conservation law in §1.3. Another useful field is the *enthalpy* per unit mass defined by

$$P = \epsilon + \frac{p}{\rho}. \quad (1.26)$$

The enthalpy is useful because its differential is

$$dP = d\epsilon + \frac{dp}{\rho} - \frac{p}{\rho^2} d\rho = \frac{dp}{\rho} + T ds, \quad (1.27)$$

which implies that Euler's equation can be re-written as

$$\frac{D\mathbf{u}}{Dt} + \nabla\Phi = -\frac{\nabla p}{\rho} = -\nabla P + T\nabla s. \quad (1.28)$$

For barotropic flows the terms involving the entropy  $s$  in (1.27) and (1.28) are absent and thus in this case the pressure force is irrotational.

### 1.2.3 The polytropic fluid model

The polytropic fluid model is a special case of a barotropic fluid with a power law dependence of pressure on density, i.e.,  $p \propto \rho^\gamma$  for some constant  $\gamma$ . For example, in the case of homentropic flow of an ideal gas  $\gamma$  would be the usual ratio of specific heats. Also, this model includes the standard shallow-water equations if  $\gamma = 2$  and  $\rho$  is identified with the layer depth as a function of horizontal coordinates (cf. §1.6 below). We will use this model frequently in this book and therefore we summarize its mechanical structure here.

We write the pressure as

$$p(\rho) = \frac{\rho_0 c_0^2}{\gamma} \left(\frac{\rho}{\rho_0}\right)^\gamma \quad (1.29)$$

where  $\rho_0$  is a reference density and  $c_0$  is the linear sound speed for perturbations around  $\rho = \rho_0$ . This is consistent with the nonlinear definition of

the sound speed as  $c^2 = \partial p / \partial \rho$  where the derivative is taken at constant entropy  $s$ . The advantage of writing the polytropic pressure in the form (1.29) is that it keeps separate the effects of varying  $\gamma$  and of varying the linear sound speed, which is instructive in applications.

The corresponding specific internal energy  $\epsilon(\rho)$  is

$$\epsilon(\rho) = \frac{c_0^2}{\gamma(\gamma - 1)} \left( \frac{\rho}{\rho_0} \right)^{\gamma-1} \quad (1.30)$$

and the specific enthalpy is

$$P = \frac{c_0^2}{\gamma - 1} \left( \frac{\rho}{\rho_0} \right)^{\gamma-1} = \gamma \epsilon. \quad (1.31)$$

The governing momentum equation is then given by (1.28) without the  $\nabla s$  term.

### 1.3 Conservation laws and energy

The continuity equation (1.4) is an example of a *local conservation law*, which expresses the conservation of mass with the scalar density  $\rho$  and the flux vector  $\rho \mathbf{u}$ . In this special case the flux is purely advective, i.e., the flux equals the density times the velocity. Using (1.4), Euler's equation (1.22) can also be written in conservation form for the momentum, namely

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + p \boldsymbol{\delta}) = -\rho \nabla \Phi \quad (1.32)$$

where  $\boldsymbol{\delta}$  is the unit tensor. Thus, in the absence of the potential force (1.32) is a conservation law for the momentum vector with density  $\rho \mathbf{u}$  and flux tensor  $\rho \mathbf{u} \mathbf{u} + p \boldsymbol{\delta}$ . In this case the advective momentum flux  $\rho \mathbf{u} \mathbf{u}$  is augmented by the non-advective momentum flux  $p \boldsymbol{\delta}$  due to the pressure. The useful identity

$$\rho \frac{D\phi}{Dt} = \frac{\partial(\rho \phi)}{\partial t} + \nabla \cdot (\rho \phi \mathbf{u}) \quad (1.33)$$

makes clear that any local conservation law with non-advective flux  $\mathbf{F}$ , say, can be written in the two equivalent forms

$$\frac{D\phi}{Dt} + \frac{1}{\rho} \nabla \cdot \mathbf{F} = 0 \quad \text{or} \quad \frac{\partial(\rho \phi)}{\partial t} + \nabla \cdot (\rho \phi \mathbf{u} + \mathbf{F}) = 0. \quad (1.34)$$

For  $\phi = 1$  and  $\phi = \mathbf{u}$  this yields the two forms of the mass and momentum conservation laws, respectively.