GENERAL NOTATION.

THE axes of x, y are taken in the ecliptic of 1850·0 and the centre of mass of the earth and the moon is supposed to describe a fixed ellipse around the sun in this plane. As it is more convenient to use the motion of the centre of the earth than of this centre of mass, a slight well-known change, noted below, is necessary in the disturbing function.

The axis of x is parallel to the line joining the earth and the sun.

In the scheme of notation which follows, two sets of constants are given for the mean distance, eccentricity and sine of half the inclination of the moon's orbit. The first set is that which I have used in the expressions for the rectangular coordinates of the moon; the second set is that of Delaunay in the final form which he gives to the expressions for the longitude, latitude and parallax. The longitudes of a *planet*, of its perigee and of its node are as usual reckoned along the ecliptic to its node and then along the orbit.

	Moon	Earth	Planet
True long.	V	V'	V''
Mean long.	$w_1 = l + g + h$	T	P
Mean anom.	$w_1 - w_2 = l$	$l' = T - \varpi'$	$l'' = P - \varpi''$
Mean long. of node	$w_3 = h$	0	h''
Mean motion	n	n'	$\frac{dP}{dt}$
Mean distance	a, a	a'	a''
Eccentricity	e, e	e'	e''
Sine half inclin.	k, γ	0	γ''
Coors., origin earth	x, y, z, r		ξ, η, ζ, Δ
,, ,, sun		x', y', z', r'	x'', y'', z'', r''

$$n = \text{mean motion of the moon} = \frac{dw_1}{dt},$$

$$b_2 = \text{ ,, ,, its perigee} = \frac{dw_2}{dt},$$

$$b_3 = \text{ ,, ,, ,, node} = \frac{dw_3}{dt},$$

c_1, c_2, c_3 are the canonical constants complementary to w_1, w_2, w_3 *after* the problem of the moon's motion as disturbed by the sun, supposed to move in a fixed elliptic orbit, has been solved.

R = the disturbing function of this problem, arising from the direct attraction of a planet.

The symbols for the mean longitudes of the planets are: Mercury, Q; Venus*, V; Mars, M; Jupiter, J; Saturn, S.

* No confusion will be caused by the use of the same symbol for the true longitude of the moon and the mean longitude of Venus. The notation of Radau has been adopted with a few changes.

SECTION I.

THE EQUATIONS OF VARIATIONS.

LET w_1, w_2, w_3 represent the mean longitudes of the moon, of its perigee and of its node, and suppose the problem of the moon as disturbed by the sun, has been solved. Then it is well known that if a disturbing function R be added to the force function of the moon's motion, the change in the latter due to R can be obtained by solving the equations

$$(1) \quad \frac{dc_i}{dt} = \frac{\partial R}{\partial w_i}, \quad \frac{dw_i}{dt} = -\frac{\partial R}{\partial c_i} + b_i \qquad (i = 1, 2, 3),$$

where b_1, b_2, b_3 are the mean motions of the moon, of its perigee and of its node, and c_1, c_2, c_3 are the canonical constants corresponding to w_1, w_2, w_3; the c_i are functions of the arbitrary constants n, e, k of the moon's motion, and they also contain the constants n', e', depending on the sun's motion. The substitution of the new values of c_i, w_i, thus found, in the expressions for the coordinates will give the disturbed position of the moon at any time.

The constant part of R only gives constant additions to the b_i, i.e., to the mean motions*: this part will be neglected, since it has no effect on the new terms to be found. Hence $b_1 = n$.

Change to the semi-canonical system n, c_2, c_3, retaining the w_i unchanged. Putting

$$\frac{dc_1}{dn} = -a^2\beta,$$

and remembering that

$$(1a) \quad \frac{dc_1}{dc_2} = -\frac{db_2}{dn}, \quad \frac{dc_1}{dc_3} = -\frac{db_3}{dn}, \quad \frac{db_2}{dc_3} = \frac{db_3}{dc_2},$$

* I have found these changes in an earlier paper: *Trans. Amer. Math. Soc.*, Vol. v. pp. 279—284. A fresh computation just made gives 2″·69, −1″·42, for the mean motions of the perigee and node respectively. The former is 0″·03 more than the value given in the paper.

1—2

we obtain equations (1) in the semi-canonical form

$$(2)\quad \begin{cases} \dfrac{dn}{dt} = \dfrac{1}{a^2\beta}\left(-\dfrac{\partial R}{\partial w_1} - \dfrac{db_2}{dn}\cdot\dfrac{dc_2}{dt} - \dfrac{db_3}{dn}\cdot\dfrac{dc_3}{dt}\right), & \dfrac{dw_1}{dt} = \dfrac{1}{a^2\beta}\dfrac{\partial R}{\partial n} + b_1, \\ \dfrac{dc_2}{dt} = \dfrac{\partial R}{\partial w_2}, & \dfrac{dw_2}{dt} = -\dfrac{\partial R}{\partial c_2} + b_2 + \left(\dfrac{dw_1}{dt} - b_1\right)\dfrac{db_2}{dn}, \\ \dfrac{dc_3}{dt} = \dfrac{\partial R}{\partial w_3}, & \dfrac{dw_3}{dt} = -\dfrac{\partial R}{\partial c_3} + b_3 + \left(\dfrac{dw_1}{dt} - b_1\right)\dfrac{db_3}{dn}; \end{cases}$$

in these equations b_2, b_3, c_1 are supposed to be expressed in terms of n, c_2, c_3 and R in terms of n, c_2, c_3, w_1, w_2, w_3.

Consider any periodic term of R:

$$R = n'^2 \mathrm{a}^2 A \cos(qt + q') = n'^2 \mathrm{a}^2 A \cos(i_1 w_1 + i_2 w_2 + i_3 w_3 + q''t + q'''),$$

where a is the linear constant of Hill's variational orbit and of my lunar theory and A is a numerical coefficient (that is, its dimensions with respect to time, space and mass are zero); $q''t + q'''$ is a combination of the solar and planetary arguments. Then since $qt + q'$ is independent of the c_i and A of the w_i, the first three of equations (2) become

$$\frac{dn}{dt} = \frac{n'^2}{\beta}\cdot\frac{\mathrm{a}^2}{a^2} A \frac{dq}{dn}\sin(qt + q'),$$

$$\frac{dc_2}{dt} = -i_2 n'^2 \mathrm{a}^2 A \sin(qt + q'),$$

$$\frac{dc_3}{dt} = -i_3 n'^2 \mathrm{a}^2 A \sin(qt + q').$$

It will now be supposed that R contains a small factor whose square may be neglected. The coefficients in the right-hand members of the last set will then be constants and we can integrate. Put $m = \dfrac{n'}{n}$, and let δn, δc_2, δc_3 denote the increments of n, c_2, c_3, due to R. Then

$$(3)\quad \begin{cases} \dfrac{\delta n}{n} = -\dfrac{m}{\beta}\cdot\dfrac{\mathrm{a}^2}{a^2}\cdot\dfrac{dq}{dn}\dfrac{n'}{q} A\cos(qt + q'), \\ \dfrac{\delta c_2}{na^2} = i_2 m\,.\,\dfrac{\mathrm{a}^2}{a^2}\dfrac{n'}{q} A\cos(qt + q'), \\ \dfrac{\delta c_3}{na^2} = i_3 m\,.\,\dfrac{\mathrm{a}^2}{a^2}\dfrac{n'}{q} A\cos(qt + q'). \end{cases}$$

Again, if we put

$$A_1 = \frac{n}{\mathrm{a}^2}\frac{d}{dn}(\mathrm{a}^2 A),$$

$$A_2 = -a^2 n \frac{dA}{dc_2},$$

$$A_3 = -a^2 n \frac{dA}{dc_3},$$

the other three of equations (2) become

$$(3\text{a})\quad \begin{cases} \dfrac{dw_1}{dt} = \dfrac{n'^2}{n\beta} \cdot \dfrac{\mathrm{a}^2}{a^2} A_1 \cos(qt+q') + b_1, \\[2mm] \dfrac{dw_2}{dt} = \dfrac{n'^2}{n} \cdot \dfrac{\mathrm{a}^2}{a^2}\left(A_2 + \dfrac{A_1}{\beta} \cdot \dfrac{db_2}{dn}\right) \cos(qt+q') + b_2, \\[2mm] \dfrac{dw_3}{dt} = \dfrac{n'^2}{n} \cdot \dfrac{\mathrm{a}^2}{a^2}\left(A_3 + \dfrac{A_1}{\beta} \cdot \dfrac{db_3}{dn}\right) \cos(qt+q') + b_3. \end{cases}$$

Let δb_1, δb_2, δb_3, δw_1, δw_2, δw_3 denote increments due to R. Then

$$\delta b_1 = \delta n = -\frac{n'}{\beta} \cdot \frac{\mathrm{a}^2}{a^2}\frac{dq}{dn} \cdot \frac{n'}{q} A \cos(qt+q'),$$

$$\delta b_2 = \frac{db_2}{dn}\delta n + \frac{db_2}{dc_2}\delta c_2 + \frac{db_2}{dc_3}\delta c_3$$

$$= n'\frac{\mathrm{a}^2}{a^2} \cdot \frac{n'}{q} A\left(-\frac{1}{\beta} \cdot \frac{db_2}{dn} \cdot \frac{dq}{dn} + a^2\frac{db_2}{dc_2} \cdot i_2 + a^2\frac{db_2}{dc_3} \cdot i_3\right)\cos(qt+q'),$$

by equations (1a).

Putting $$q_2 = -na^2\frac{dq}{dc_2}, \quad q_3 = -na^2\frac{dq}{dc_3},$$

and using the last of equations (1a), we find

$$\delta b_2 = -n'\frac{\mathrm{a}^2}{a^2} \cdot \frac{n'}{q} A\left(\frac{1}{\beta} \cdot \frac{db_2}{dn} \cdot \frac{dq}{dn} + q_2\right)\cos(qt+q');$$

similarly, $$\delta b_3 = -n'\frac{\mathrm{a}^2}{a^2} \cdot \frac{n'}{q} A\left(\frac{1}{\beta} \cdot \frac{db_3}{dn} \cdot \frac{dq}{dn} + q_3\right)\cos(qt+q').$$

The undisturbed values of b_1, b_2, b_3 are of course equal to those of $\dfrac{dw_1}{dt}$, $\dfrac{dw_2}{dt}$, $\dfrac{dw_3}{dt}$, respectively. To obtain δw_1, δw_2, δw_3 (the increments of w_1, w_2, w_3), it is necessary to replace w_i by $w_i + \delta w_i$, and b_i by $b_i + \delta b_i$ in (3a), to substitute for δb_i the values just obtained, and then to integrate. These operations give

$$(4)\quad \begin{cases} \delta w_1 = \dfrac{1}{\beta}\dfrac{\mathrm{a}^2}{a^2}\left(m\dfrac{n'}{q}A_1 - \dfrac{dq}{dn}\dfrac{n'^2}{q^2}A\right)\sin(qt+q'), \\[2mm] \delta w_2 = \dfrac{\mathrm{a}^2}{a^2}\left\{m\dfrac{n'}{q}\left(A_2 + \dfrac{A_1}{\beta}\dfrac{db_2}{dn}\right) - \dfrac{n'^2}{q^2}A\left(\dfrac{q_2}{n} + \dfrac{1}{\beta}\dfrac{db_2}{dn}\dfrac{dq}{dn}\right)\right\}\sin(qt+q'), \\[2mm] \delta w_3 = \dfrac{\mathrm{a}^2}{a^2}\left\{m\dfrac{n'}{q}\left(A_3 + \dfrac{A_1}{\beta}\dfrac{db_3}{dn}\right) - \dfrac{n'^2}{q^2}A\left(\dfrac{q_3}{n} + \dfrac{1}{\beta}\dfrac{db_3}{dn}\dfrac{dq}{dn}\right)\right\}\sin(qt+q'). \end{cases}$$

The equations (3), (4) constitute the solution of the problem.

The form in which a periodic term of R arises (see Section II.) is

$$R = \frac{1}{4}\frac{m''}{m'}n'^2\mathrm{a}^2 A\cos(qt+q').$$

It is therefore necessary to multiply the right-hand members of (3), (4) by $m''/4m'$.

Next, let

s' = no. of seconds in the daily mean motion of the sun = 3548″·19,

s = " " " " argument $qt+q'$.

Then $$\frac{n'}{q}=\frac{s'}{s}.$$

Also, put for brevity

$$(5)\quad \begin{cases} f=\dfrac{1}{4}\dfrac{m''}{m'}\dfrac{\mathrm{a}^2}{a^2}\dfrac{s'^2}{\beta}206265, \\ f'=\dfrac{1}{4}\dfrac{m''}{m'}\dfrac{\mathrm{a}^2}{a^2}\dfrac{ms'}{\beta}206265. \end{cases}$$

The coefficients of the right-hand members being thus expressed in seconds of arc, equations (3), (4) become

$$(6)\quad \begin{cases} \dfrac{\delta n}{n}=-f'\dfrac{dq}{dn}\dfrac{A}{s}\cos(qt+q'), \\ \dfrac{\delta c_2}{na^2}=i_2\beta f'\dfrac{A}{s}\cos(qt+q'), \\ \dfrac{\delta c_3}{na^2}=i_3\beta f'\dfrac{A}{s}\cos(qt+q'), \\ \delta w_1=\left(f'\dfrac{A_1}{s}-f\dfrac{A}{s^2}\dfrac{dq}{dn}\right)\sin(qt+q'), \\ \delta w_2=\left\{\dfrac{f'}{s}\left(\beta A_2+\dfrac{db_2}{dn}A_1\right)-\dfrac{f}{s^2}A\left(\beta\dfrac{q_2}{n}+\dfrac{db_2}{dn}\cdot\dfrac{dq}{dn}\right)\right\}\sin(qt+q'), \\ \delta w_3=\left\{\dfrac{f'}{s}\left(\beta A_3+\dfrac{db_3}{dn}A_1\right)-\dfrac{f}{s^2}A\left(\beta\dfrac{q_3}{n}+\dfrac{db_3}{dn}\cdot\dfrac{dq}{dn}\right)\right\}\sin(qt+q'); \end{cases}$$

where, to recall certain definitions,

$$\beta=-\frac{1}{a^2}\cdot\frac{dc_1}{dn},\quad q=i_1n+i_2b_2+i_3b_3+q'',$$

$$q_2=-na^2\frac{dq}{dc_2},\quad q_3=-na^2\frac{dq}{dc_3}.$$

Numerical form of the equations of variations.

It remains to be seen how these quantities may be put into numerical form. In f, f' the factor $\dfrac{\mathrm{a}^2}{a^2}$ is immediately obtained from Hill's results* for the variational orbit; m is a well known quantity; $\dfrac{m''}{m'}$ is known as soon as

* *Amer. Jour. Math.*, Vol. I. p. 249.

the particular planet is chosen; thus f, f' remain the same for a given planet and f/f' for all planets. The coefficients A, A_1, A_2, A_3 will be found later on, while s is known as soon as the particular term of R has been chosen.

There remain for calculation

$$(7) \quad \frac{dc_1}{dn}, \ \frac{db_2}{dn}, \ \frac{db_3}{dn}, \ \frac{db_2}{dc_2}, \ \frac{db_2}{dc_3} = \frac{db_3}{dc_2}, \ \frac{db_3}{dc_3},$$

which, depending only on the orbit of the moon as attracted by the sun and earth, are the same for every perturbation of the moon's motion, and therefore apply not only to the present investigation but also to all investigations where a disturbing function R is added to the moon's force function.

Some idea of the degree of accuracy required is desirable. The largest known inequality is that with the argument $l + 16T - 18V$, which has a coefficient of about 15″. For this $i_1 = 1$, $i_2 = -1$, $i_3 = 0$. The principal part is given by

$$-f\frac{A}{s^2}\frac{dq}{dn} = -f\frac{A}{s^2}\left(1 - \frac{db_2}{dn}\right) = -f\frac{A}{s^2}(1 + \cdot 01486).$$

There is no other coefficient which is so great as 2″. Since the degree of accuracy aimed at is 0″·01, it will be sufficient to use four place logarithms and four significant figures for the functions (7) so that the final results will be accurate to at least three significant figures. But certain of the functions are only needed to one or two significant figures, as will appear immediately.

The functions c_1, c_2, c_3 are the same as Delaunay's L, $G - L$, $H - G$ after the final transformations and the changes to his final system of arbitraries, n, e, γ have been made. As my results will be used for the calculation of the moon functions, it will be more convenient to transfer A_2, A_3 to my constants e, k.

Let $\frac{dQ}{dn}$ denote the derivative of a function Q with respect to n when it is expressed in terms of n, c_2, c_3, and $\left(\frac{dQ}{dn}\right)$ when it is expressed in terms of n, e^2, γ^2. Then the following equations serve for the transformation of the derivatives of Q from one set to the other*:

$$(8) \quad \begin{cases} \dfrac{dQ}{dn} = \left(\dfrac{dQ}{dn}\right) - \dfrac{dQ}{dc_2} \cdot \left(\dfrac{dc_2}{dn}\right) - \dfrac{dQ}{dc_3}\left(\dfrac{dc_3}{dn}\right), \\ \dfrac{dQ}{dc_2} = \left[\dfrac{dQ}{de^2} - \dfrac{dQ}{dc_3}\dfrac{dc_3}{de^2}\right] \div \dfrac{dc_2}{de^2}, \\ \dfrac{dQ}{dc_3} = \left[\dfrac{dQ}{d\gamma^2} - \dfrac{dQ}{dc_2} \cdot \dfrac{dc_2}{d\gamma^2}\right] \div \dfrac{dc_3}{d\gamma^2}; \end{cases}$$

* The functions considered here involve e, γ only in the even powers.

the same equations will serve for the transformation from the set n, c_2, c_3 to the set n, e², k² if we replace e^2, γ^2 by e², k², respectively.

It is first to be noticed that

$$\begin{aligned}
c_2 &= na^2e^2 \left\{-\tfrac{1}{2} + \text{power series in } m,\ e^2,\ \gamma^2,\ e'^2,\ \left(\frac{a}{a'}\right)^2\right\},\\
c_3 &= na^2\gamma^2 \{-2 + \quad ,, \quad ,, \quad ,, \quad ,, \quad \},\\
c_1 &= na^2 \{\ 1 + \quad ,, \quad ,, \quad ,, \quad ,, \quad \},\\
b_2 &= nm^2 \{\ \tfrac{3}{4} + \quad ,, \quad ,, \quad ,, \quad ,, \quad \},\\
b_3 &= nm^2 \{-\tfrac{3}{4} + \quad ,, \quad ,, \quad ,, \quad ,, \quad \};
\end{aligned}$$

the power series in each case vanishing with m. Hence when for Q in equations (8) is put c_1, b_2 or b_3, the principal part of the first term in each equation will have a portion independent of e^2, γ^2, while the other terms will have one of these quantities as a factor. Since $e = \frac{1}{18}$, $\gamma = \frac{1}{22}$ approximately, two significant figures will be fully sufficient for the values of $\frac{dc_2}{dn}$, $\frac{dc_3}{dn}$. It is to be noted that the second and third of equations (8) do not depend on derivatives with respect to n, and therefore that we may change to the variables e, γ or to e, k, from c_2, c_3 without reference to the first equation.

From Newcomb's transformation* of Delaunay's values for L, G, H, we have

Terms in	$\frac{c_2}{na^2e^2}$	$\frac{1}{a^2e^2}\left(\frac{dc_2}{dn}\right)$	Terms in	$\frac{c_3}{na^2\gamma^2}$	$\frac{1}{a^2\gamma^2}\left(\frac{dc_3}{dn}\right)$
m^0	− ·50015	+ ·1667	m^0	− 1·99704	+ ·6657
m^2	+ ·01686	− ·0393	m^2	− ·00322	+ ·0075
m^3	+ ·00644	− ·0215	m^3	+ ·00049	− ·0016
m^4	+ ·00170	− ·0074	m^4	+ ·00005	− ·0002
rem.	+ ·00032	− ·0020	rem.	− ·00022	+ ·0010
Sum	− ·47483	+ ·0965	Sum	− 1·99994	+ ·6724

The remainders in the first and third columns are obtained from the values of c_2, c_3 (calculated by processes independent of the nature of convergence along powers of m) given by myself†; those in the second and fourth columns are estimated from them. In any case the results for $\left(\frac{dc_2}{dn}\right)$, $\left(\frac{dc_3}{dn}\right)$ are correct within two per cent. and this is all the accuracy necessary for our purpose.

In a similar manner the values of $\left(\frac{db_2}{dn}\right)$, $\left(\frac{db_3}{dn}\right)$ can be obtained with

* "Action of the Planets on the Moon," *Amer. Eph. Papers*, Vol. v. Pt 3, pp. 201, 202.
† "Theory of the Motion of the Moon," *Trans. R. A. S.*, Vol. LVII. pp. 64, 65.

Cambridge University Press
978-1-107-66473-9 - The Inequalities in the Motion of the Moon due to the Direct Action of the Planets: An Essay which Obtained the Adams Prize in the University of Cambridge for the Year 1907
Ernest W. Brown
Excerpt
More information

more than needful accuracy from the series in powers of m given by Delaunay*, Hill†, and Adams‡ with my numerical values§. The derivatives with respect to e^2, γ^2 are obtained immediately from the last named reference. The derivatives with respect to e^2, k^2 can be immediately derived by inserting the values of $\frac{e}{\mathrm{e}}$, $\frac{\gamma}{\mathrm{k}}$ which I have also given§: the change in the derivatives with respect to n is insensible.

I find the following results to four places in the logarithms, to each of which 10 has been added:

$$\left(\frac{dc_2}{dn}\right) = +[6{\cdot}4637]\, a^2, \quad \frac{dc_2}{d\mathrm{e}^2} = -[9{\cdot}0775]\, na^2, \quad \frac{dc_2}{d\mathrm{k}^2} = -[7{\cdot}5002]\, na^2,$$

$$\left(\frac{dc_3}{dn}\right) = +[7{\cdot}1313]\, a^2, \quad \frac{dc_3}{d\mathrm{e}^2} = +[6{\cdot}7577]\, na^2, \quad \frac{dc_3}{d\mathrm{k}^2} = -[10{\cdot}3039]\, na^2,$$

$$\left(\frac{db_2}{dn}\right) = -[8{\cdot}1709], \quad \frac{db_2}{d\mathrm{e}^2} = -[7{\cdot}3974]\, n, \quad \frac{db_2}{d\mathrm{k}^2} = -[8{\cdot}7004]\, n,$$

$$\left(\frac{db_3}{dn}\right) = +[7{\cdot}5736], \quad \frac{db_3}{d\mathrm{e}^2} = -[7{\cdot}4731]\, n, \quad \frac{db_3}{d\mathrm{k}^2} = +[7{\cdot}8692]\, n.$$

Whence, accurately to four figures,

$$\frac{db_2}{dn} = -[8{\cdot}1720], \quad a^2\frac{db_2}{dc_2} = +[8{\cdot}3175], \quad a^2\frac{db_2}{dc_3} = +[8{\cdot}3960],$$

$$\frac{db_3}{dn} = +[7{\cdot}5733], \quad a^2\frac{db_3}{dc_2} = +[8{\cdot}3960], \quad a^2\frac{db_3}{dc_3} = -[7{\cdot}5698].$$

We have also

$$\frac{\mathrm{a}^2}{a^2} = +[9{\cdot}99921], \quad m = +[8{\cdot}87391], \quad \beta = +[9{\cdot}51801],$$

$$f = [22{\cdot}29358]\frac{m''}{m'}, \quad f' = [17{\cdot}61748]\frac{m''}{m'}, \quad \frac{f'}{f} = [5{\cdot}32390],$$

β being found accurately in Section III. From these quantities the terms depending on $\frac{1}{s^2}$ in (4) can be found.

Further, by substituting A_2, A_3 successively for Q in the second and third of equations (8), and putting e for e, and k for γ, we find

$$A_2 = -a^2 n\frac{dA}{dc_2} = [11{\cdot}5819]\frac{dA}{d\mathrm{e}} - [8{\cdot}4242]\frac{dA}{d\mathrm{k}},$$

$$A_3 = -a^2 n\frac{dA}{dc_3} = [10{\cdot}7440]\frac{dA}{d\mathrm{k}} - [8{\cdot}7782]\frac{dA}{d\mathrm{e}}.$$

* *Comptes Rendus*, Vol. LXXIV. pp. 19 *et sqq.*
† *Acta Math.*, Vol. VIII. pp. 1—36. *Annals of Math.*, Vol. IX. pp. 31—41.
‡ *Monthly Notices, R. A. S.*, Vol. XXXVIII. pp. 43—49.
§ *Loc. cit.*, Vol. LXIV. p. 532.

Finally, making these various substitutions in (3), (4), the equations of variations become

$$(9)\quad \begin{cases} \dfrac{\delta n}{n} = (-i_1 + \cdot 01486 i_2 - \cdot 003744 i_3) f' \dfrac{A}{s} \cos(qt+q'), \\ \dfrac{\delta c_2}{na^2} = +[9\cdot 51801]\, i_2 f' \dfrac{A}{s} \cos(qt+q'), \\ \dfrac{\delta c_3}{na^2} = +[9\cdot 51801]\, i_3 f' \dfrac{A}{s} \cos(qt+q'); \\ \delta w_1 = \left\{(-i_1 + \cdot 01486 i_2 - \cdot 003744 i_3) f \dfrac{A}{s^2} + f' \dfrac{A_1}{s}\right\} \sin(qt+q'), \\ \delta w_2 = \left\{(+\cdot 01486 i_1 - \cdot 007066 i_2 - \cdot 008148 i_3) f \dfrac{A}{s^2} \right. \\ \qquad \left. + \left(-[8\cdot 1720] A_1 + [11\cdot 0999] \dfrac{dA}{de} - [7\cdot 9422] \dfrac{dA}{dk}\right) \dfrac{f'}{s}\right\} \sin(qt+q'), \\ \delta w_3 = \left\{(-\cdot 003744 i_1 - \cdot 008148 i_2 + \cdot 001210 i_3) f \dfrac{A}{s^2} \right. \\ \qquad \left. + \left(+[7\cdot 5733] A_1 + [10\cdot 2620] \dfrac{dA}{dk} - [8\cdot 2962] \dfrac{dA}{de}\right) \dfrac{f'}{s}\right\} \sin(qt+q'), \end{cases}$$

where $$A_1 = \frac{n}{a^2} \frac{d}{dn} (A a^2),$$

A being expressed in terms of n, c_2, c_3; the figures enclosed in square brackets being, as elsewhere, logarithms with 10 added.

These equations replace the equations of variations for δl, δg, δh, δa, δe, $\delta\gamma$ first given by G. W. Hill* and calculated also by Radau†. To reduce the above system to theirs it is necessary to find δc_2, δc_3 in terms of δn, δe, $\delta\gamma$. But it is simple to compare the first terms of δw_1, on which the principal part of each long-period inequality depends. Radau finds

$$\delta w_1 = \{(-3\cdot 0576 i + \cdot 05601 i' - \cdot 01124 i'') p + \dots\} P \sin(qt+q'),$$

where $i_1 = i + i' + i''$, $\quad i_2 = i' + i''$, $\quad i_3 = i''$, $\quad pP = \beta f \dfrac{A}{s^2}$.

This reduction gives

$$\delta w_1 = \left\{(-\cdot 9894 i_1 + \cdot 01476 i_2 - \cdot 003705 i_3) f \frac{A}{s^2} + \dots\right\} \sin(qt+q'),$$

which is about $\frac{1}{100}$ less than my value. But I have been unable to find the coefficient $-3\cdot 0576 i$ in δw_1, which Radau gives: from his data I make this coefficient $-3\cdot 0791$‡ which, reduced to my form, becomes $-\cdot 9964$. The

* *American Eph. Papers*, Vol. III. p. 390.

† *Loc. cit.*, pp. 35, 36.

‡ This apparent error seems to be due to some confusion in the substitution of the numerical values for n before and after the final Delaunay transformation.