THEORY

OF

DIFFERENTIAL EQUATIONS.

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PART IV.

PARTIAL DIFFERENTIAL EQUATIONS.

BY

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PREFACE.

THESE two volumes, now published as Part IV of the present work, are my final contribution towards the fulfilment of a promise made twenty-one years ago. They are devoted to the theory of partial differential equations.

Though the work thus is completed, no claim is made that every topic of importance has been discussed. In the earlier volumes, indications of omissions from other portions of the whole subject were given and need not now be repeated: here also, there have been definite omissions. Nothing, for instance, is said concerning the researches of Picard and Dini on the method of successive approximations for the construction of an integral which obeys assigned conditions; these investigations limit the variables to real values, and throughout the treatise I have dealt with variables having complex values. Formal questions, such as those which arise out of the application of the theory of groups, are hardly mentioned; here, as in the preceding volumes, I have concerned myself with organic properties, given by applications of the theory of functions, rather than with formal properties. Again, the subject of boundary problems is not dealt with; it appears to me to belong to the theory of functions in its applications to mathematical physics rather than to the theory of differential equations. In the branches of the subject that are discussed, I have tried to deal as completely as possible with what seems to me to be essential: and I have omitted what are purely formal extensions, to equations of general order, of the properties of equations of the second order when such extensions contain no intrinsic novelty.

In the preparation of the volumes, I have consulted the works of many writers; and references are freely given. My aim has been to make these references relate to the main issues; not a few results, extracted from memoirs, have been used to construct examples; and the name of the author is (I hope) given in every such case. But I have not attempted to select and arrange the references, so that they might make the framework of a history of the subject; had the latter been my purpose, names such as Lagrange, Cauchy, Jacobi, whose work is now the common possession of all writers, would have received more frequent specific references in my pages. It will be seen that Darboux's treatise, *Théorie générale des surfaces*, and Goursat's three volumes, *Leçons sur l'intégration des équations aux dérivées partielles*, have been frequently quoted: I wish to make also a comprehensive acknowledgement of my indebtedness to those works.

The earlier of the two volumes is devoted mainly to equations of the first order. The theory of these equations may be regarded as almost complete, because the actual integration of the equations is made to depend solely upon the solution of difficulties which occur in connection with a system of ordinary equations of the first order.

An introduction to the subject is provided by Cauchy's existence-theorem; it is discussed in the first two chapters. The next chapter is specially concerned with linear equations and linear systems; these admit of a separate and special mode of treatment. The fourth chapter gives an exposition of what, on the whole, I regard as the most effective method of integration for non-linear equations: it contains what is usually called Jacobi's second method, with Mayer's developments. In the succeeding chapter will be found Lagrange's classification of integrals, based upon the process of variation of parameters: but something still remains to be done in this branch of the subject, because even simple examples shew that the customary classes may fail to be entirely comprehensive. The next three chapters are devoted to Cauchy's method of characteristics, alike for two and for any number of independent variables, and to the geometrical associations in the case of two independent variables. Then follows a chapter dealing with Lie's methods, based upon contact-transformations and upon the properties of groups of functions: it was possible to abbreviate this chapter, because Pfaff's problem had already been discussed in the first volume of this work. A chapter has been added dealing with the equations of theoretical dynamics, partly because of their intrinsic connection with partial equations, yet mainly in order to shew the origin of what is usually called Jacobi's first method of integration of partial equations. The concluding chapter of this volume discusses those simultaneous equations of the first order, involving more than one dependent variable, which can be integrated by operations of the same class as those in any of the methods mentioned.

The later of these two volumes is devoted to the consideration of partial equations of the second order and of higher orders, mainly (though not entirely) involving two independent variables. A perusal of the volume will shew that, outside the limits of Cauchy's existence-theorem, knowledge is fragmentary: the inversion of operations of the second order has not yet been discovered and, accordingly, any effective process consists of a succession of operations of the first order.

After a chapter devoted to the discussion of questions connected with the existence of integrals and, in particular, to the discussion of the constitution of a general integral, two chapters are occupied with Laplace's method (and with its developments, due to Darboux) for the integration of the homogeneous linear equations of the second order: the effective success of the method depends upon the vanishing of some invariant, in one or other of two progressively constructed sets of functions involving the coefficients of the original equation. The result raises the question of the form of equations, the primitive of which can be expressed in finite terms: and, to this matter, one chapter is assigned.

In the attempt to integrate any equation of the second order, it is natural to enquire whether an equation of the first order exists which is its complete equivalent: and equations, characterised by this property, will obviously constitute a distinct class. Such, indeed, were the equations of the second order for which integrals (now called intermediate) were first obtained; and one method of their construction is due to Monge. Later, another (and a more direct) method for their construction was given by Boole: but both methods assume that a special form attaches to the intermediate integral, and the assumption demands that a very restricted form shall be possessed by the original equation. Basing his argument entirely upon an assumed type of integral, Ampère devised another process of integration: his method makes no demand for the existence of an intermediate integral: and the result is often effective when no such integral exists. All these three methods, (and another method of some generality, as given), require the construction of integrable combinations of one (and ultimately the same) set of subsidiary equations, when they are applied to the same original equation. But Ampère's method is applicable also to equations of less restricted form.

It may, however, happen that an equation of the second order is not of the restricted form or, being of that form, does not possess an intermediate integral, or is not amenable to Ampère's method. In that case, a method due to Darboux may be applicable, whereby a compatible equation of the second order (or of some higher order) can be constructed; provided only that a compatible equation of finite order can be obtained, a primitive of the original equation can be derived. To these matters, three chapters are given: they explain the working processes that are effective for the determination of an integral in finite terms, whether by a single equation or a set of equations.

One chapter is devoted to the generalisation of integrals which involve some arbitrary parameters, and another to the discussion of characteristics of equations of the second order. The investigations in both of these chapters are clearly incomplete: they could be continued along lines that lead to the complete classification of integrals of equations of the first order.

In the theory of equations of the first order, much information is given by Lie's general theory of contact-transformations: and an obvious investigation is thereby suggested as to whether there is a corresponding theory for equations of order higher than the first. The question has been considered, and partly solved, by Bäcklund and others: one chapter gives an outline of their work: it is clear that much yet remains to be done in this subject.

In the last three chapters of the volume, some of the preceding methods and theories are extended to equations, which are of order higher than the second or which involve more than two independent variables. Only the simplest extensions are discussed: they could be amplified to any extent: but the result would be merely an accumulation of formal theorems possessing neither individuality nor intrinsic value.

From this brief sketch of the contents of these two volumes, it will be manifest that, in the theory of equations of order higher than the first, there are many gaps and that the theory is far from complete: and even a summary perusal of the volumes will give some indication of these gaps. It is my intention to point out, in a presidential address which will be delivered to the London Mathematical Society next month, some of the more obvious and practicable questions which are waiting for solution. Of these, there is no lack: it is only the workers who are wanted.

On not a few occasions, it has been my privilege to acknowledge the help which has been given to me by the Staff of the University Press. Once more, an opportunity comes to me: and I gladly seize it, to express my indebtedness to them all for the care, the attention, and the consideration, by which they have lightened what to me is never an easy or a simple duty.

So I pass from a task, which has filled the greater part of many years of my life, which has broadened in my view as they passed, and which has suffered interruptions that threatened to end it before its completion. Many of its defects are known to me: after it has gone from me, others will become apparent. Nevertheless, my hope is that my work will ease the labour of those who, coming after me, may desire to possess a systematic account of this branch of pure mathematics.

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