CHAPTER I.

INTRODUCTION: TWO EXISTENCE-THEOREMS.

1. THE investigations, which constitute this Part of the present work, are devoted to the consideration of properties of partial differential equations. In text-books which deal with the modes of constructing the integrals of such equations, several processes are given, often with the main purpose of obtaining the integrals in finite terms; but the processes are limited in the scope of their application, because the equations which prove amenable to their action are few in character and not infrequently have been artificially constructed. When these processes either are not applicable or cannot conveniently be completed, no information concerning the solution of the equation would then be obtained; indeed, they offer no guarantee that an integral even exists.

Accordingly, it is desirable to discuss the whole theory of partial differential equations from the foundations and, in the course of that discussion, not only to revise known results but also, so far as may be possible, to place them in their fitting positions in the ordered body of doctrine. Such a discussion was found to be necessary for the proper establishment of results relating to ordinary differential equations. It is even more necessary in the case of partial differential equations, partly because the inversion of simultaneous partial differential operations is more difficult than the inversion of ordinary differential operations, partly because the suggestions as to the character of an integral, as offered by processes of inversion, are less significant for partial equations than for ordinary equations.

2. Two kinds of illustration should suffice for a justification of this last statement.

One mode of attempting to discover the character of the most complete integral of a partial equation would be by generalisation from the case of an ordinary equation.

For an ordinary equation, which has y for its dependent variable and x for its independent variable, the integral is made complete by the assignment of initial values to the variables; that is, y is some function of x and so, when a constant value is assigned to x, the function y and all its derivatives become constants. As the equation is to be satisfied and yet the integral is to be as complete as possible, these constants will be as unrestricted as possible: and therefore it is to be expected that some at least of them will be arbitrary constants. There thus arises a suggestion that the most complete integral will be such that, when some constant value is assigned to x, the function y and some of its derivatives acquire arbitrary constant values. The suggested property has been established under appropriate limitations and conditions.

To extend these results, if possible, to partial differential equations, consider a single partial equation of the first order, having z for its dependent variable and $x_1, \ldots, x_n$ for its independent variables. If an integral exists, that integral must determine z as a function of $x_1, \ldots, x_n$; and so, when an initial value a_n is assigned to x_n, the first derivatives of z with respect to $x_1, \ldots, x_{n-1}$ can be deduced from an assigned expression for z, and then (save in special circumstances) the partial equation determines the first derivative with regard to x_n. By using the equation in combination with the expressions for the first derivatives, the derivatives of higher order can be obtained for the value a_n of x_n; and thus no limitation appears to be imposed on the value of z as an assigned function of $x_1, \ldots, x_{n-1}$, when $x_n = a_n$. If the integral is to be as general as possible, it is reasonable to expect that the assigned function shall be as general as possible. But at this stage, questions arise as to what is the most general function admissible? Is it to be made general by possessing the greatest possible number of arbitrary constants? Can the assigned function be an arbitrary function, subject possibly to limitations imposed by the partial equation? and, if so, must it be explicit or may it be given implicitly, for example, by means of quadratures which cannot be effected in finite terms? Or are all the modes indicated for securing the generality of the integral admissible, so that there are different kinds of general integrals? and if so, are there any relations among the various integrals? To such questions the argument offers no hint of an answer.

Similarly, when a partial equation of the second order is propounded in the same variables z, $x_1, \ldots, x_n$, the extension of the results obtained for ordinary equations suggests that z and $\dfrac{\partial z}{\partial x_n}$ should acquire assigned values as functions of $x_1, \ldots, x_{n-1}$, when $x_n = a_n$. For the values of $\dfrac{\partial z}{\partial x_1}, \ldots, \dfrac{\partial z}{\partial x_{n-1}}$, when $x_n = a_n$, could be deduced from the value of z; and then the values of $\dfrac{\partial^2 z}{\partial x_r \partial x_s}$, for $r = 1, \ldots, n-1$, and $s = 1, \ldots, n$, could be deduced from the values of $\dfrac{\partial z}{\partial x_1}, \ldots, \dfrac{\partial z}{\partial x_n}$ already known; and the partial equation would (save in special circumstances) determine the value of $\dfrac{\partial^2 z}{\partial {x_n}^2}$. As before, the values thus obtained, when combined with the use of the partial equation, lead to the values of all the derivatives. Thus all the quantities associated with z are known: at the utmost, only special limitations appear to be imposed upon the assigned functions by the process adopted; and therefore it is reasonable to expect that the integral will become the most general possible when the two assigned functions are as general as possible. Again, at this stage, questions arise as to the constitution of the generality of these assigned functions. Is the generality to be secured, by arranging that they shall involve the greatest possible number of arbitrary constants? or by making them independent arbitrary functions of $x_1, \ldots, x_{n-1}$? or by associating them with a possibly even more general function of $x_1, \ldots, x_n$ for the particular value a_n of x_n? If the functions are arbitrary, must they be given explicitly or may they be given implicitly as, for example, by uncompleted quadratures? Again, are all the modes admissible as alternatives, so that they lead to different kinds of general integrals? and if so, what relations (if any) subsist among the integrals? As in the former case, the argument offers no hint of answer to the questions.

3. In both the instances that have been briefly considered, the argument offers suggestions and even stirs expectations: that this is the limit of the attention to be paid to it, can perhaps be most simply seen by a particular case. Applied to a couple of simultaneous partial equations determining a couple of dependent variables, it would lead to a suggestion that the most general integral would involve at least two sets of general elements, whatever be their form; yet the integral of the simultaneous equations

$$\left.\begin{aligned}\frac{\partial^2 z_1}{\partial x^2} - a^2\frac{\partial^2 z_1}{\partial y^2} + 2a\frac{\partial z_1}{\partial y} + \frac{\partial z_2}{\partial x} - a\frac{\partial z_2}{\partial y} + z_2 &= f(x, y)\\ \frac{\partial z_1}{\partial x} + a\frac{\partial z_1}{\partial y} - z_1 + z_2 &= g(x, y)\end{aligned}\right\}$$

is

$$\left.\begin{aligned}z_1 &= f(x, y) - g(x, y) - \frac{\partial g(x, y)}{\partial x} + a\frac{\partial g(x, y)}{\partial y}\\ z_2 &= f(x, y) - \frac{\partial f(x, y)}{\partial x} - a\frac{\partial f(x, y)}{\partial y}\\ &\quad + 2a\frac{\partial g(x, y)}{\partial y} + \frac{\partial^2 g(x, y)}{\partial x^2} - a^2\frac{\partial^2 g(x, y)}{\partial y^2}\end{aligned}\right\},$$

a being a constant; manifestly it contains no arbitrary element. In fact, the utmost to be inferred from the argument is that some kinds of equations may possess integrals involving arbitrary elements in their most general forms, and that there may be different kinds of general integrals.

Whether these general integrals include all the integrals of an equation is a matter that demands separate consideration, to be undertaken later in another line of inquiry: and, naturally, a detailed consideration of the generality of integrals must also be undertaken later.

4. Another mode of attempting to discover the character of the most complete integral of a partial equation consists in comparing differential equations, constructed from initial integral equations, with those integral equations: but it is easily seen to be untrustworthy.

Thus if an integral equation

$$\frac{az + b}{cz + d} = \phi(x_1, \ldots, x_n),$$

where $ad - bc = 1$, be propounded, the result of eliminating the constants between the equation and its derivatives leads to the set of partial equations

$$\frac{1}{p_1}\frac{\partial\phi}{\partial x_1} = \frac{1}{p_2}\frac{\partial\phi}{\partial x_2} = \ldots = \frac{1}{p_n}\frac{\partial\phi}{\partial x_n},$$

where

$$p_r = \frac{\partial z}{\partial x_r},$$

for $r = 1, \ldots, n$. The process cannot be reversed, so as to lead to an inference that the most general integral of the set of partial equations contains three arbitrary essential constants: the inference would be incorrect, for the set of equations is satisfied by

$$f(z) = \phi(x_1, \ldots, x_n),$$

where $f(z)$ is any function of z containing any number of arbitrary constants.

Again, if there is given an integral equation

$$f(x_1, \ldots, x_n, z, a_1, \ldots, a_m) = 0,$$

it is possible to construct a set of partial equations with which the integral equation is consistent, by forming the n derived equations which give the values of $\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}$, and then eliminating the m constants $a_1, \ldots, a_m$. For the present purpose, the m constants may be assumed to be not reducible to a smaller number, and m may be assumed not greater than n; also, when elimination takes place, the number of resulting equations will be not less than $n + 1 - m$. It will be assumed that the number of such equations in the set is actually $n + 1 - m$; each of them is partial, and of the first order.

If m is equal to n, there is a single partial equation: and the argument suggests that a single partial equation of the first order may possess an integral involving n arbitrary constants: it does not prove this result, for there is nothing to shew that the partial equation is not of a special form, arising from the limitation that it has been deduced from an integral equation of specified form. The argument offers no contribution to the question as to whether, if the integral is possessed by the partial equation, it is the most general integral.

If m is less than n, there is a set of simultaneous partial equations of the first order: and the argument might be held to suggest that $n+1-m$ simultaneous partial equations of the first order, involving one dependent variable and n independent variables, may possess a common integral involving m constants. The result is, of course, not proved and it is not true in general fact: for, independently of the impossibility of reversing the process of elimination, the $n+1-m$ equations are affected by the form of the original integral and therefore will have relations with one another, while such a set of partial equations postulated initially need not have any relations with one another. Thus the existence of a common integral is even more doubtful than in the case of a single equation: if it exists, no inference as to its generality can be drawn.

5. After these explanations and criticisms, it is manifest that attempts to obtain information as to the solution of partial equations by vague extensions of the knowledge of the solution of ordinary equations must be abandoned. The constructive process, that will be adopted instead of them, consists in the gradual establishment of results, beginning with the proof of the existence of integrals possessing definite assigned characters. The actual construction of the integrals when their existence has once been established, the discussion of the range of their generality, and the possibility of using them in the derivation of integrals of other kinds, all are matters for subsequent investigation.

It will be assumed that, save in special examples, the number of independent variables is n; and they will usually be denoted by $x_1, \ldots, x_n$. The number of dependent variables may be taken as m, the simplest case arising when $m=1$; they will be denoted by $z_1, \ldots, z_m$; and when there is only one variable, it will be denoted by z. For the present purpose, these dependent variables are to be determined by partial differential equations; let the number of such equations in a given set be s, and suppose that the highest derivatives that occur in them are of order μ.

Let derivatives of each of the equations be constructed, of all orders up to those of order κ inclusive. Then the total number of equations in the amplified set is

$$\begin{aligned} &s\,\{1+n+\tfrac{1}{2}n(n+1)+\ldots \text{ to } (\kappa+1) \text{ terms}\} \\ &= s\,\frac{(n+1)(n+2)\ldots(n+\kappa)}{1\,.\,2\,.\ldots\,\kappa} \\ &= sN, \end{aligned}$$

say; and the total number of dependent quantities, being the dependent variables and their derivatives of all orders up to $\mu + \kappa$ inclusive, is (or can be, for some of the dependent quantities may not occur explicitly)

$$m\{1 + n + \tfrac{1}{2}n(n+1) + \ldots \text{ to } (\kappa + \mu + 1) \text{ terms}\}$$
$$= m\frac{(n+1)(n+2)\ldots(n+\kappa+\mu)}{1 \, . \, 2 \, . \ldots . \, (\kappa+\mu)}$$
$$= mNK,$$

where

$$K = \frac{(n+\kappa+1)\ldots(n+\kappa+\mu)}{(\kappa+1)\ldots(\kappa+\mu)}.$$

The factor K is obviously always greater than unity; and therefore if $s < m$, or if $s = m$, the number sN is less than mNK. The number of equations in the amplified set is less than the number of dependent quantities in the amplified aggregate; and therefore it will generally be impossible to eliminate the dependent quantities from among the equations. Were such elimination possible, the results would take the form of relations between the independent variables: and these, of course, do not occur. There is therefore nothing incompatible with the analytical nature of the case, if $s < m$, or if $s = m$.

Next, consider the possible hypothesis that $s > m$. The factor K is greater than unity; but its value decreases as κ increases, and it tends towards unity with large increase of κ. Let κ_1 be the earliest value of κ for which

$$K < \frac{s}{m};$$

then for the value κ_1, and for every value of κ which is greater than κ_1, we have

$$s > mK,$$

and therefore

$$sN > mNK.$$

For such values of κ, the number of equations in the amplified system is greater than the number of dependent quantities in the amplified aggregate. The dependent quantities could then, in general, be eliminated from the amplified system of equations; the results would take the form of relations among the independent variables alone, and such relations cannot occur. Such a conclusion is, in general, not compatible with the nature of the case: and therefore, in general, s cannot be greater than m. If however the elimination could be performed for any given set of equations, amplified in the manner indicated, the final relations would be evanescent, and the incompatibility would not appear. This last event could occur only if such conditions were satisfied by the original system and consequent conditions were satisfied by the amplified system, as would reduce the number of independent equations in the amplified system so that, at the utmost, it should not be larger than the number of dependent quantities in the amplified aggregate.

Hence, in general, the number of equations in a given system must not be greater than the number of dependent variables involved; but the number of equations may be the greater in particular systems, and the investigation of the necessary and sufficient conditions will be a matter for subsequent discussion.

It is clear without detailed argument that, when s is less than m and when the equations are general, then $m-s$ of the dependent variables can have values assigned (either quite arbitrarily or arbitrarily within proper limits), still leaving as many equations as undetermined dependent variables.

Accordingly, the most general case to be considered for the present is that in which the number of equations is the same as the number of dependent variables.

6. Two properties of such a system of equations may be mentioned; their importance is mainly formal, and only a brief consideration is needed.

The first of the properties can be stated as follows: if a system of m partial equations in m dependent variables involves derivatives of order higher than the first, it can be replaced by an equivalent system of equations containing only derivatives of the first order, the number of independent equations in the new system being the same as the number of dependent variables which it involves.

The property is practically obvious and so hardly requires proof: it can be seen in connection with any particular example. Let there be a single equation, involving derivatives of the second order as the highest: when n new dependent variables are introduced by the equations

$$\frac{\partial z}{\partial x_r} = p_r, \qquad (r = 1, \ldots, n),$$

the given equation can be expressed in a form

$$f\left(x_1, \ldots, x_n, z, p_1, \ldots, p_n, \frac{\partial p_1}{\partial x_1}, \ldots, \frac{\partial p_n}{\partial x_n}\right) = 0,$$

which involves only derivatives of the first order; and the new system now contains $n+1$ equations, involving $n+1$ dependent variables with derivatives of the first order.

It may be added that the main use of the property lies in deducing existence-theorems for equations of order higher than the first from the existence-theorems which soon will be established for systems of equations of the first order.

An extended form of the property enables us, not merely to replace any given system by a system containing only derivatives of the first order, but also to secure that each equation, which in the new system involves derivatives of the first order, is linear in those derivatives. Thus, in the preceding example, additional dependent variables would be introduced by the equations

$$\frac{\partial z}{\partial x_\mu} = p_\mu, \qquad \frac{\partial p_\mu}{\partial x_s} = q_{\mu s},$$

for μ and $s = 1, \ldots, n$: the original equation takes the form of a relation

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n, q_{11}, \ldots, q_{nn}) = 0$$

among the variables free from derivatives; and the derived equations

$$\frac{\partial f}{\partial x_s} + p_s \frac{\partial f}{\partial z} + \sum_{\mu=1}^{n} \frac{\partial f}{\partial p_\mu} q_{\mu s} + \sum_{\lambda, \mu}\sum \frac{\partial f}{\partial q_{\lambda\mu}} \frac{\partial q_{\lambda\mu}}{\partial x_s} = 0$$

are formed for $s = 1, \ldots, n$. All the equations are linear in the derivatives which are of the first order; but it should be noted that the number of equations in the modified system is larger than the number of dependent variables, though the conditions for coexistence are satisfied.

When the number of variables is other than very few, the extended form of the property tends to be cumbrous. It is, however, of definite use, as will be seen later (Chap. XVIII), as part of a method for obtaining integrals of equations of order higher than the first when they possess integrals that are expressible in finite terms.

7. The other of the two properties indicated in § 6 bases the solution of a system of m partial equations in m dependent variables upon the solution of one equation (or of more than one equation) in a single dependent variable in association with algebraic processes: the substituted equation or equations usually (but not universally) involve derivatives of order higher than those which occurred in the original system.

Reverting to the method adopted in § 5, and applying it for the purpose of eliminating $z_2, \ldots, z_m$ and all their derivatives, we should have mN equations in the amplified set, while the number of dependent quantities to be eliminated is $(m-1)NK$. Accordingly, let κ_1 be the least value of κ for which

$$K < \frac{m}{m-1},$$

and therefore

$$(m-1)NK < mN.$$

The dependent quantities, composed of $z_2, \ldots, z_m$ and their derivatives, can be eliminated from the amplified set of equations: the results of the elimination will take the form of one equation or more than one equation involving z_1 and its derivatives, the latter being of order higher than those which occur in the original system. Moreover, by the algebraic processes, all the dependent quantities that are eliminated are expressible in terms of those that survive. Accordingly, when the solution of the equation or equations in z_1 is known, the other dependent quantities can be regarded as known: and then the solution of the original system will have been obtained.

It should be added that this property is not of importance in the general theory: its chief value lies in the fact that it provides a method which sometimes is effective in leading to the solution of particular classes of equations.

Preparation for Cauchy's Theorem: the first of the subsidiary Existence-theorems.

8. We proceed now to the establishment of some positive results, in particular, to the establishment of Cauchy's theorem affirming the existence of integrals of a system of partial equations.