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## Understanding Probability

*Understanding Probability* is a unique and stimulating approach to a first course in probability. The first part of the book demystifies probability and uses many wonderful probability applications from everyday life to help the reader develop a feel for probabilities. The second part, covering a wide range of topics, teaches clearly and simply the basics of probability.

This fully revised Third Edition has been packed with even more exercises and examples, and it includes new sections on Bayesian inference, Markov chain Monte Carlo simulation, hitting probabilities in random walks and Brownian motion, and a new chapter on continuous-time Markov chains with applications. Here you will find all the material taught in an introductory probability course. The first part of the book, with its easy-going style, can be read by anybody with a reasonable background in high school mathematics. The second part of the book requires a basic course in calculus.

HENK TIJMS is Emeritus Professor at Vrije University in Amsterdam. He is the author of several textbooks and numerous papers on applied probability and stochastic optimization. In 2008 Henk Tijms received the prestigious INFORMS Expository Writing Award for his publications and books.

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Third Edition

HENK TIJMS  
*Vrije University, Amsterdam*



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## Preface

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Why do so many students find probability difficult? Could it be the way the subject is taught in so many textbooks? When I was a student, a class in topology made a great impression on me. The teacher asked us not to take notes during the first hour of his lectures. In that hour, he explained ideas and concepts from topology in a non-rigorous, intuitive way. All we had to do was listen in order to grasp the concepts being introduced. In the second hour of the lecture, the material from the first hour was treated in a mathematically rigorous way and the students were allowed to take notes. I learned a lot from this approach of interweaving intuition and formal mathematics.

This book is written very much in the same spirit. It first helps you develop a “feel for probabilities” before presenting the more formal mathematics. The book is not written in a theorem–proof style. Instead, it aims to teach the novice the concepts of probability through the use of motivating and insightful examples. No mathematics are introduced without specific examples and applications to motivate the theory. Instruction is driven by the need to answer questions about probability problems that are drawn from real-world contexts. The book is organized into two parts. Part One is informal, using many thought-provoking examples and problems from the real world to help the reader understand what probability really means. Probability can be fun and engaging, but this beautiful branch of mathematics is also indispensable to modern science. The basic theory of probability – including topics that are usually not found in introductory probability books – is covered in Part Two. Designed for an introductory probability course, this can be read independently of Part One. The book can be used in a variety of disciplines ranging from finance and engineering to mathematics and statistics. As well as for introductory courses, the book is also suitable for self-study. The prerequisite knowledge for Part Two is a basic course in calculus, but Part One can be read by anybody with a reasonable background in high school mathematics.

This book distinguishes itself from other introductory probability texts by its emphasis on why probability works and how to apply it. Simulation in

interaction with theory is the perfect instrument for clarifying and enlivening the basic concepts of probability. For this reason, computer simulation is used to give insights into such key concepts as the law of large numbers, which come to life through the results of many simulation trials. The law of large numbers and the central limit theorem lie at the center in the first part of the book. Many of the examples used to illustrate these themes deal with lotteries and casino games.

Good exercises are an essential part of each textbook. Much care has been paid to the collecting of problems that will enhance your understanding of probability. You will be challenged to think about ideas, rather than simply plug numbers into formulas. As you work through them, you may find that probability problems are often harder than they first appear. It is particularly true for the field of probability that a feeling for the subject can only be acquired by going through the process of learning-by-doing.

*New to this edition*

This edition continues the well-received structure of the book, where Part One is informal and gives many probability applications from daily life, while Part Two teaches clearly and simply the mathematics of probability theory. Revisions and additions are in particular made in Part Two in order to serve better the requirements for classroom use of the book. The most significant additions are:

- more material on coincidences in gambling games (Chapters 3 and 4)
- hitting probabilities in random walks and Brownian motion with applications to gambling (Chapters 3 and 5)
- a new section on optimal stopping (Chapter 3)
- more material on Kelly betting (Chapter 5)
- a new section on Bayesian statistics for the continuous case (Chapter 13)
- a new section on the law of the iterated logarithm (Chapter 14)
- a new section on Markov chain Monte Carlo simulation (Chapter 15)
- a new chapter on continuous-time Markov chains (Chapter 16).

Also, many new exercises are added. A detailed solution manual is available from the author to instructors who adopt this book as a course text.

Finally, thanks to all those who have made valuable comments on the text, particularly Ted Hill (Georgia Institute of Technology), Karl Sigman (Columbia University), Virgil Stokes (Uppsala University) and my colleagues Rein Nobel, Ad Ridder and Aart de Vos at the Vrije University.