LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor M. Reid, Mathematics Institute,
University of Warwick, Coventry CV4 7AL, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press
at www.cambridge.org/mathematics

259 Models and computability, S.B. COOPER & J. TRUSS (eds.)
260 Groups St Andrews 1997 in Bath I, C.M. CAMPBELL *et al.* (eds.)
261 Groups St Andrews 1997 in Bath II, C.M. CAMPBELL *et al.* (eds.)
262 Analysis and logic, C.W. HENSON, J. IOVINO, A.S. KECHRIS & E. ODELL
263 Singularity theory, W. BRUCE & D. MOND (eds.)
264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds.)
265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
267 Surveys in combinatorics, 1999, J.D. LAMB & D.A. PREECE (eds.)
268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
269 Ergodic theory and topological dynamics of group actions on homogeneous spaces, M.B. BEKKA & M. MAYER
271 Singular perturbations of differential operators, S. ALBEVERIO & P. KURASOV
272 Character theory for the odd order theorem, T. PETERFALVI. Translated by R. SANDLING
273 Spectral theory and geometry, E.B. DAVIES & Y. SAFAROV (eds.)
274 The Mandelbrot set, theme and variations, T. LEI (ed.)
275 Descriptive set theory and dynamical systems, M. FOREMAN, A.S. KECHRIS, A. LOUVEAU & B. WEISS (eds.)
276 Singularities of plane curves, E. CASAS-ALVERO
277 Computational and geometric aspects of modern algebra, M. ATKINSON *et al.* (eds.)
278 Global attractors in abstract parabolic problems, J.W. CHOLEWA & T. DLOTKO
279 Topics in symbolic dynamics and applications, F. BLANCHARD, A. MAASS & A. NOGUEIRA (eds.)
280 Characters and automorphism groups of compact Riemann surfaces, T. BREUER
281 Explicit birational geometry of 3-folds, A. CORTI & M. REID (eds.)
282 Auslander–Buchweitz approximations of equivariant modules, M. HASHIMOTO
283 Nonlinear elasticity, Y.B. FU & R.W. OGDEN (eds.)
284 Foundations of computational mathematics, R. DEVORE, A. ISERLES & E. SÜLI (eds.)
285 Rational points on curves over finite fields, H. NIEDERREITER & C. XING
286 Clifford algebras and spinors (2nd Edition), P. LOUNESTO
287 Topics on Riemann surfaces and Fuchsian groups, E. BUJALANCE, A.F. COSTA & E. MARTÍNEZ (eds.)
288 Surveys in combinatorics, 2001, J.W.P. HIRSCHFELD (ed.)
289 Aspects of Sobolev-type inequalities, L. SALOFF-COSTE
290 Quantum groups and Lie theory, A. PRESSLEY (ed.)
291 Tits buildings and the model theory of groups, K. TENT (ed.)
292 A quantum groups primer, S. MAJID
293 Second order partial differential equations in Hilbert spaces, G. DA PRATO & J. ZABCZYK
294 Introduction to operator space theory, G. PISIER
295 Geometry and integrability, L. MASON & Y. NUTKU (eds.)
296 Lectures on invariant theory, I. DOLGACHEV
297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
298 Higher operads, higher categories, T. LEINSTER (ed.)
299 Kleinian groups and hyperbolic 3-manifolds, Y. KOMORI, V. MARKOVIC & C. SERIES (eds.)
300 Introduction to Möbius differential geometry, U. HERTRICH-JEROMIN
301 Stable modules and the D(2)-problem, F.E.A. JOHNSON
302 Discrete and continuous nonlinear Schrödinger systems, M.J. ABLOWITZ, B. PRINARI & A.D. TRUBATCH
303 Number theory and algebraic geometry, M. REID & A. SKOROBOGATOV (eds.)
304 Groups St Andrews 2001 in Oxford I, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds.)
305 Groups St Andrews 2001 in Oxford II, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds.)
306 Geometric mechanics and symmetry, J. MONTALDI & T. RATIU (eds.)
307 Surveys in combinatorics 2003, C.D. WENSLEY (ed.)
308 Topology, geometry and quantum field theory, U.L. TILLMANN (ed.)
309 Corings and comodules, T. BRZEZINSKI & R. WISBAUER
310 Topics in dynamics and ergodic theory, S. BEZUGLYI & S. KOLYADA (eds.)
311 Groups: topological, combinatorial and arithmetic aspects, T.W. MÜLLER (ed.)
312 Foundations of computational mathematics, Minneapolis 2002, F. CUCKER *et al.* (eds.)
313 Transcendental aspects of algebraic cycles, S. MÜLLER-STACH & C. PETERS (eds.)
314 Spectral generalizations of line graphs, D. CVETKOVIĆ, P. ROWLINSON & S. SIMIĆ
315 Structured ring spectra, A. BAKER & B. RICHTER (eds.)
316 Linear logic in computer science, T. EHRHARD, P. RUET, J.-Y. GIRARD & P. SCOTT (eds.)
317 Advances in elliptic curve cryptography, I.F. BLAKE, G. SEROUSSI & N.P. SMART (eds.)
318 Perturbation of the boundary in boundary-value problems of partial differential equations, D. HENRY
319 Double affine Hecke algebras, I. CHEREDNIK
320 L-functions and Galois representations, D. BURNS, K. BUZZARD & J. NEKOVÁŘ (eds.)
321 Surveys in modern mathematics, V. PRASOLOV & Y. ILYASHENKO (eds.)
322 Recent perspectives in random matrix theory and number theory, F. MEZZADRI & N.C. SNAITH (eds.)
323 Poisson geometry, deformation quantisation and group representations, S. GUTT *et al.* (eds.)
324 Singularities and computer algebra, C. LOSSEN & G. PFISTER (eds.)
325 Lectures on the Ricci flow, P. TOPPING
326 Modular representations of finite groups of Lie type, J.E. HUMPHREYS
327 Surveys in combinatorics 2005, B.S. WEBB (ed.)

328 Fundamentals of hyperbolic manifolds, R. CANARY, D. EPSTEIN & A. MARDEN (eds.)
329 Spaces of Kleinian groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds.)
330 Noncommutative localization in algebra and topology, A. RANICKI (ed.)
331 Foundations of computational mathematics, Santander 2005, L.M PARDO, A. PINKUS, E. SÜLI & M.J. TODD (eds.)
332 Handbook of tilting theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds.)
333 Synthetic differential geometry (2nd Edition), A. KOCK
334 The Navier–Stokes equations, N. RILEY & P. DRAZIN
335 Lectures on the combinatorics of free probability, A. NICA & R. SPEICHER
336 Integral closure of ideals, rings, and modules, I. SWANSON & C. HUNEKE
337 Methods in Banach space theory, J.M.F. CASTILLO & W.B. JOHNSON (eds.)
338 Surveys in geometry and number theory, N. YOUNG (ed.)
339 Groups St Andrews 2005 I, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds.)
340 Groups St Andrews 2005 II, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds.)
341 Ranks of elliptic curves and random matrix theory, J.B. CONREY, D.W. FARMER, F. MEZZADRI & N.C. SNAITH (eds.)
342 Elliptic cohomology, H.R. MILLER & D.C. RAVENEL (eds.)
343 Algebraic cycles and motives I, J. NAGEL & C. PETERS (eds.)
344 Algebraic cycles and motives II, J. NAGEL & C. PETERS (eds.)
345 Algebraic and analytic geometry, A. NEEMAN
346 Surveys in combinatorics 2007, A. HILTON & J. TALBOT (eds.)
347 Surveys in contemporary mathematics, N. YOUNG & Y. CHOI (eds.)
348 Transcendental dynamics and complex analysis, P.J. RIPPON & G.M. STALLARD (eds.)
349 Model theory with applications to algebra and analysis I, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds.)
350 Model theory with applications to algebra and analysis II, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds.)
351 Finite von Neumann algebras and masas, A.M. SINCLAIR & R.R. SMITH
352 Number theory and polynomials, J. MCKEE & C. SMYTH (eds.)
353 Trends in stochastic analysis, J. BLATH, P. MÖRTERS & M. SCHEUTZOW (eds.)
354 Groups and analysis, K. TENT (ed.)
355 Non-equilibrium statistical mechanics and turbulence, J. CARDY, G. FALKOVICH & K. GAWEDZKI
356 Elliptic curves and big Galois representations, D. DELBOURGO
357 Algebraic theory of differential equations, M.A.H. MACCALLUM & A.V. MIKHAILOV (eds.)
358 Geometric and cohomological methods in group theory, M.R. BRIDSON, P.H. KROPHOLLER & I.J. LEARY (eds.)
359 Moduli spaces and vector bundles, L. BRAMBILA-PAZ, S.B. BRADLOW, O. GARCÍA-PRADA & S. RAMANAN (eds.)
360 Zariski geometries, B. ZILBER
361 Words: Notes on verbal width in groups, D. SEGAL
362 Differential tensor algebras and their module categories, R. BAUTISTA, L. SALMERÓN & R. ZUAZUA
363 Foundations of computational mathematics, Hong Kong 2008, F. CUCKER, A. PINKUS & M.J. TODD (eds.)
364 Partial differential equations and fluid mechanics, J.C. ROBINSON & J.L. RODRIGO (eds.)
365 Surveys in combinatorics 2009, S. HUCZYNSKA, J.D. MITCHELL & C.M. RONEY-DOUGAL (eds.)
366 Highly oscillatory problems, B. ENGQUIST, A. FOKAS, E. HAIRER & A. ISERLES (eds.)
367 Random matrices: High dimensional phenomena, G. BLOWER
368 Geometry of Riemann surfaces, F.P. GARDINER, G. GONZÁLEZ-DIEZ & C. KOUROUNIOTIS (eds.)
369 Epidemics and rumours in complex networks, M. DRAIEF & L. MASSOULIÉ
370 Theory of *p*-adic distributions, S. ALBEVERIO, A.YU. KHRENNIKOV & V.M. SHELKOVICH
371 Conformal fractals, F. PRZYTYCKI & M. URBAŃSKI
372 Moonshine: The first quarter century and beyond, J. LEPOWSKY, J. MCKAY & M.P. TUITE (eds.)
373 Smoothness, regularity and complete intersection, J. MAJADAS & A. G. RODICIO
374 Geometric analysis of hyperbolic differential equations: An introduction, S. ALINHAC
375 Triangulated categories, T. HOLM, P. JØRGENSEN & R. ROUQUIER (eds.)
376 Permutation patterns, S. LINTON, N. RUŠKUC & V. VATTER (eds.)
377 An introduction to Galois cohomology and its applications, G. BERHUY
378 Probability and mathematical genetics, N. H. BINGHAM & C. M. GOLDIE (eds.)
379 Finite and algorithmic model theory, J. ESPARZA, C. MICHAUX & C. STEINHORN (eds.)
380 Real and complex singularities, M. MANOEL, M.C. ROMERO FUSTER & C.T.C. WALL (eds.)
381 Symmetries and integrability of difference equations, D. LEVI, P. OLVER, Z. THOMOVA & P. WINTERNITZ (eds.)
382 Forcing with random variables and proof complexity, J. KRAJÍČEK
383 Motivic integration and its interactions with model theory and non-Archimedean geometry I, R. CLUCKERS, J. NICAISE & J. SEBAG (eds.)
384 Motivic integration and its interactions with model theory and non-Archimedean geometry II, R. CLUCKERS, J. NICAISE & J. SEBAG (eds.)
385 Entropy of hidden Markov processes and connections to dynamical systems, B. MARCUS, K. PETERSEN & T. WEISSMAN (eds.)
386 Independence-friendly logic, A.L. MANN, G. SANDU & M. SEVENSTER
387 Groups St Andrews 2009 in Bath I, C.M. CAMPBELL *et al.* (eds.)
388 Groups St Andrews 2009 in Bath II, C.M. CAMPBELL *et al.* (eds.)
389 Random fields on the sphere, D. MARINUCCI & G. PECCATI
390 Localization in periodic potentials, D.E. PELINOVSKY
391 Fusion systems in algebra and topology, M. ASCHBACHER, R. KESSAR & B. OLIVER
392 Surveys in combinatorics 2011, R. CHAPMAN (ed.)
393 Non-abelian Fundamental Groups and Iwasawa Theory, J.COATES *et al.* (eds.)
394 Variational problems in differential geometry, R. BIELAWSKI, K. HOUSTON & M. SPEIGHT (eds.)
395 How Groups Grow, A. MANN

London Mathematical Society Lecture
Note Series: 395

How Groups Grow

AVINOAM MANN
Einstein Institute of Mathematics
Hebrew University of Jerusalem

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107657502

First published 2012

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-65750-2 Paperback

Cambridge University Press
978-1-107-65750-2 - How Groups Grow
Avinoam Mann
Frontmatter
More information

Contents

Cambridge University Press
978-1-107-65750-2 - How Groups Grow
Avinoam Mann
Frontmatter
More information

Preface

The topic of growth entered group theory, with a geometric motivation, at the middle of last century. It associates to each finitely generated group a number-theoretical function, its *growth*, and investigates the relationship between the properties of the group and of its growth function. The subject has attracted attention gradually, until, in about 1980, two seminal theorems were proved: first Misha Gromov determined the groups of polynomial growth, and a short time later Slava Grigorchuk constructed groups of intermediate growth. Both theorems were the starting points for rich mathematical developments. As far as I know, there is no detailed treatment of growth in book form. Of course, I should not ignore Pierre de la Harpe's remarkable text *Topics in Geometric Group Theory*, a large part of which is devoted to growth, but in that text most results are quoted without proofs (an exception is the chapter about Grigorchuk's group, but even about that group we were able to include in the present text more recent results).

For several years now, I have been teaching courses devoted to one or the other of the above results. Most of these were one-semester courses in the Einstein Institute of Mathematics in the Hebrew University, but some were given at several Italian universities. The present notes were prepared for, and based on, these courses. They can be divided roughly into four main parts: introductory, polynomial growth, intermediate growth, and miscellany. The first part includes the first two chapters, of which the first consists mainly of definitions and examples, and sets the context for the rest of the book. The second chapter was written mainly for the benefit of readers who are either non-specialists, or beginners, in group theory, but Section 2.2 of that chapter, which develops the elementary theory of growth, should be read carefully. Section 2.5 is a digression about isoperimetric inequalities. Then follow seven chapters devoted to polynomial growth. Chapter 3 discusses groups of linear growth, for which a completely elementary treatment is available, and indeed preceded Gromov's theorem; the rest of the book is independent of that chapter. Then come three chapters about the growth of nilpotent, soluble, and linear groups. Chapter 4 includes a classification, up to commensurability, of nilpotent groups of small growth degree, and Chapter 5 includes a reduction of the proof of the uniformity of exponential growth of soluble groups to the polycyclic case. While most of the notes are essentially self-contained, Chapter 6, on linear groups, consists mostly of results that are quoted without proof. Then we introduce asymptotic cones, and finally, in Chapter 8 we prove Gromov's theorem. The approach is similar to Gromov's original one, as modified by Wilkie and van den Dries, and Gromov himself. In the definition of asymptotic cones we were influenced by Cornelia Drutu's nice survey [Dr 02], but of course we had to supply the details. It seems that this is the first detailed treatment of asymptotic cones in a book form. A different approach to Gromov's theorem was suggested recently by Bruce Kleiner, applying harmonic functions on Cayley graphs. While this has the advantage of not relying on the deep solution of Hilbert's fifth problem, the method is very different from the ones of this book, and we refer to it only in passing. Chapter 9 derives corollaries of Gromov's theorem for groups that are infinitely generated, but are of locally polynomial growth. The readers, or instructors, who want to arrive quickly at Gromov's theorem will take what they need from the first two chapters, and skip Chapter 3 and Sections 4.2 and 5.2.

The next four chapters deal with intermediate growth. First we construct Grigorchuk's group, following Grigorchuk's original approach, applying permutations of the unit interval. Then we derive several of the many remarkable properties of that group, in particular proof that it has intermediate growth, giving explicit lower and upper bounds. In the next chapter we describe generalizations of the construction, and also other approaches, via actions on regular finitary trees, or finite automata, and some other examples of intermediate growth. The next two chapters relate intermediate growth to two other important group theoretical notions: amenability and residual finiteness. The chapter on amenability contains further discussion of isoperimetry. If one is interested mainly in intermediate growth, it is possible to go directly from Chapter 2 to this part (except that in Chapter 12 we quote from Chapter 7 the notion of ultralimit).

The last four chapters are the "miscellany" part. We first show how to calculate the growth of, say, amalgamated products, or other group-theoretical constructs, from the growth of the components. We also give a detailed calculation of the growth of one group of geometrical interest, to demonstrate that even for groups of very easy description, such calculations are far from trivial. In the next chapter we discuss the generating growth function. We want first to point out the applicability of analytic methods. Then we prove the rationality result for finitely generated abelian groups, with a proof that relates the group-theoretical growth to a ring-theoretical one. We also treat briefly, without proofs, the growth of hyperbolic groups. The next chapter proves uniform exponential growth, with explicit bounds, for several classes of groups, amalgamated free products, HNN-extensions, and groups of positive deficiency, in particular one-relator groups. The final chapter introduces and discusses briefly conjugacy class growth, a different type of growth from the previous one, but closely related to it.

We end with a list of open research problems.

There are many topics that are not touched upon in this book; to do so would have delayed the publication considerably, perhaps indefinitely. We do not discuss connections with geometry, e.g. the relation between volume growth in a Riemannian manifold and the growth of its fundamental group, or between the volume growth of a Lie group and the growth of a discrete dense subgroup. Kleiner's proof was mentioned already. As we said, we treat uniform exponential growth for several classes of groups, but the important cases of soluble or linear groups are only quoted without proofs, except for a partial result for soluble groups. We also only quote without proofs the results by Grigorchuk and John S. Wilson, that the polynomiality assumption in Gromov's theorem can be relaxed, if we restrict ourselves to residually nilpotent, or residually soluble, groups. These, and other interesting ideas, may form the contents of another publication.

Preliminary versions of these notes have been circulating for some time, and I received comments on them from quite a few people, as well as from my students, and from an anonymous referee. My sincere thanks to all of them. And thanks to Eva Goldman for drawing the figures in Chapter 11.