

CONTENTS

CHAPTER I.

INTRODUCTION OF RECTANGULAR MATRICES AND DETERMINOIDS.

§§	PAGES
1. Rectangular matrices; derived matrices; minor matrices; corranged and deranged derived matrices; derived products; complete and incomplete derived products; steps; conjugate matrix . . .	1-5
2. Abbreviated notations for a matrix	5-12
3. Determinoid defined as an algebraical sum of complete derived products; Rule of Signs	12-17
4. Abbreviated notations for a determinoid	17-19
5. Some simple properties of determinoids	19-21

CHAPTER II.

AFFECTS OF THE ELEMENTS AND DERIVED PRODUCTS OF A MATRIX OR DETERMINOID.

6-8. Derived products of a matrix or determinoid; complete, incomplete, extended and completed derived products; affects of the elements and derived products; sign determined by an affect; ways of obtaining the affects	22-30
9-12. Changes in the affect and sign of a derived product caused by the interchange of two parallel rows in the fundamental matrix or the interchange of two suffixes of the same kind in the derived product	30-47
13. The sign of a derived product is independent of the order of arrangement of its factors	47-48
14. The affect of a derived product is equal to the number of forward moves by which the derived product can be brought into the leading position	48-50
15. Changes in the sign of a derived product caused by inversions in the orders of arrangement of the rows of the fundamental matrix or determinoid	50-52
16. Changes in the value of a determinoid due to inversions in the orders of arrangement of its rows	52-54

CHAPTER III.

SEQUENCES AND THE AFFECTS OF DERIVED SEQUENCES.

§§		PAGES
17-18.	Sequences; derived sequences; minor sequences; corranged and deranged derived sequences; complete, incomplete, extended and completed derived sequences: moves; affects of derived sequences; ways of obtaining the affects	55-62
19.	Theorems concerning the affects of derived sequences	62-71
20-21.	Change in the affect of a derived sequence caused by the interchange of two elements of the fundamental sequence either in the fundamental sequence or in the derived sequence	72-82
22.	Reduction of the affects of derived products to the affects of derived sequences	83-85

CHAPTER IV.

AFFECTS OF DERIVED MATRICES AND DERIVED DETERMINOIDS.

23.	Affects of derived matrices and derived determinoids	86-90
24.	Extended and completed derived matrices and determinoids; in extension and completion the affect remains unaltered	90-91
25.	Theorems concerning the affects of derived matrices and derived determinoids	91-102
26.	Theorems concerning the affects of complementary corranged derived matrices	102-104

CHAPTER V.

EXPANSIONS OF A DETERMINOID.

27.	Expansions of a determinoid in terms of the elements of any long row	105-109
28.	Reciprocal matrices and reciprocal determinoids	110-111
29.	Properties of the short rows of a determinoid	111-113
30.	Expansion of a determinoid in terms of its simple minor determinants	113-115
31.	Classification of simple minor matrices and simple minor determinoids: long-cut and short-cut minors; superior and inferior simple minors; reduced and unreduced orders of a simple minor	115-116
32.	Expansion of a determinoid in terms of the simple minor determinants of a given long-cut minor matrix	116-123
33-34.	Algebraical sum of superior (short-cut) simple minor determinoids of given reduced order. The sum is a numerical multiple of the fundamental determinoid	123-134
35-36.	Algebraical sum of inferior short-cut simple minor determinoids of given reduced order r ; algebraical sum of (inferior) long-cut simple minor determinoids of given reduced order r . Each sum is equal to the algebraical sum of the affected minor determinants of order r and to the algebraical sum of the affected derived products of order r	134-138

CONTENTS

ix

§§		PAGES
37–38.	Algebraical sum of the products obtained from two fixed complementary simple minor matrices by multiplying each simple minor determinoid of the one minor matrix by its co-factor lying in the other minor matrix. When both factor determinoids in each term of the product are superior simple minors of their respective minor matrices, the sum is a numerical multiple of the fundamental determinoid	138–152

CHAPTER VI.

PROPERTIES OF A PRODUCT FORMED BY A CHAIN OF
 MATRIX FACTORS.

EQUALITY OF MATRICES. ADDITION AND SUBTRACTION OF MATRICES.		
39.	Equality of matrices; identical and conventional equality	153–155
40.	Addition and subtraction of matrices; commutative and associative properties; equivalence of conventionally equal matrices	155–157
41.	Multiplication of a matrix by a scalar quantity	157
PRODUCTS OF TWO MATRICES.		
42.	Product of two matrices taken in prescribed order: active and passive rows; standard products; reduction to standard form	158–164
43.	Properties of the passive rows in a product of two matrices; partial products	164–168
44.	Properties of the active rows in a product of two matrices	168–172
45.	Other properties of a product of two matrices: equivalence of conventionally equal matrices; distributive property; the product is in general not commutative; conjugate of the product	173–177
46.	Special cases of a standard product of two matrices: one factor a unit matrix; one factor a scalar matrix; one factor a quasi-scalar matrix; products of a matrix and its conjugate reciprocal; products of a matrix and its (principal) inverse	177–184
PRODUCTS OF THREE MATRICES.		
47.	Product of three matrices taken in prescribed order; reduction to standard form; associative and distributive properties of such a product	184–188
PRODUCT FORMED BY A CHAIN OF MATRIX FACTORS.		
48–51.	Product formed by a chain of matrix factors taken in prescribed order: associative and distributive character of such a product; active and passive rows; reduction to standard form; expressions for the elements of the product matrix	188–195
52.	Properties of the passive rows; partial products	195–199
53.	Properties of the active rows	199–204
54.	Other properties of the product: equivalence of conventionally equal matrices; conjugate of the product; multiplication of the product by a scalar quantity	204–206
55.	Special cases of a standard product formed by a chain of matrix factors: one factor a zero matrix; one factor a scalar matrix	207–208

CHAPTER VII.

DETERMINOID OF A PRODUCT FORMED BY A CHAIN OF
MATRIX FACTORS.

§§		PAGES
	DETERMINOID OF A PRODUCT OF TWO MATRICES.	
56.	Determinoid of a standard product AB of two given matrices in which the efficiency is η and the passivity is r : if $r < \eta$, then $\det AB = 0$; if $r = \eta$, then $\det AB = \det A \times \det B$; if $r \not< \eta$, then $\det AB$ is an algebraical sum of products of pairs of derived determinants of order η ; rules for evaluating $\det AB$	209–219
57.	Special cases: one factor a unit matrix; one factor a scalar matrix; product of two inversely similar matrices	219–220
	DETERMINOID OF A PRODUCT FORMED BY A CHAIN OF MATRIX FACTORS.	
58–59.	Determinoid of a standard product $AB\dots ST$ formed by a chain of matrix factors in which the efficiency is η : theorems concerning the determinoid of the product; if any passivity is less than η , then $\det AB\dots ST = 0$; if no passivity is less than η , then $\det AB\dots ST$ is an algebraical sum of products of derived determinants of order η ; general formulae; progressive development of the determinoid	221–231
60.	Special cases: one factor a unit matrix; one factor a scalar matrix; numerical factor	231–232
61–63.	Expressions for the determinant of any standard product in which the two activities are equal; determinant of a standard product of three matrices in which the two activities are equal; expressions for the determinoid of any standard product in which the two activities are not equal	232–244
	GENERALISATION.	
64.	Algebraical sum of the affected minor determinants of given order k of any standard product of given matrices	244–247

CHAPTER VIII.

MATRICES OF MINOR DETERMINANTS.

65.	Complete matrices of the minor determinants of given order of a given fundamental matrix; schemes of formation; standard schemes of formation; standard matrix of the minor determinants of given order	248–253
66.	Complete matrices of the minor determinants of order k of a product $X = AB\dots ST$ formed by a chain of given matrix factors: if any passivity of the chain is less than k , then every complete matrix of the minor determinants of order k of X is a zero matrix; if no passivity of the chain is less than k , then every complete matrix of the minor determinants of order k of X is equal to the product of any set of correspondingly formed complete matrices of the minor determinants of order k of the factor matrices A, B, \dots, S, T	253–260
67.	Reciprocal and conjugate reciprocal of a standard product of square matrices	261–264

CONTENTS

xi

CHAPTER IX.

RANK OF A MATRIX AND CONNECTIONS BETWEEN
 THE ROWS OF A MATRIX.

§§		PAGES
68.	Rank of a matrix whose elements are constants; degenerate and undegenerate matrices; singular and non-singular matrices	265-269
69-70.	Connections between the rows of such a matrix; unconnected rows; theorems concerning connections between the rows	269-278
71.	Theorems concerning the rank of such a matrix; necessary and sufficient conditions that the rank may be r ; superior limit to the rank of a product of given matrices; an undegenerate square matrix employed as a standard pre-factor or post-factor does not alter the rank; equipotent matrices	278-288
72.	Rank of a product of two mutually conjugate matrices: when the factor matrices are real; when the factor matrices are undegenerate	288-289
73.	Ranks of all complete matrices of the minor determinants of any given fundamental matrix: if the fundamental matrix has rank r , the rank of every complete matrix of the minor determinants of order s is $\binom{r}{s}$ or 0 according as $s \geq r$ or $s < r$	289-294
74.	Rank of a functional matrix and connections between its rows; equipotent functional matrices	294-298

CHAPTER X.

MATRIX EQUATIONS OF THE FIRST DEGREE.

75.	Definitions: matrix equations of the first degree; general solutions; finite and infinite solutions	299-300
76.	The equations $AX=A$, $XA=A$	300-301
77-79.	The equations $AX=C$, $XB=C$, $AXB=C$ when A and B are undegenerate square matrices: each equation has in this case a unique solution which is finite; the solution is obtained by prefixing the conjugate reciprocal (or inverse) matrix of the pre-factor of X and postfixing the conjugate reciprocal (or inverse) matrix of the post-factor of X	302-309
80-83.	The equations $AX=C$, $XB=C$, $AXB=C$ in general: augmented and unaugmented matrices of the equations; reduction to irreducible equations of the same kind; the equations admit of finite solution when and only when corresponding augmented and unaugmented matrices have equal ranks; general solutions when the conditions for finite solvability are satisfied; the special equations $AX=0$, $XB=0$, $AXB=0$; infinite solutions. When the equations are irreducible and finitely solvable, they can be solved by prefixing the conjugate reciprocal of any undegenerate simple minor square matrix of the pre-factor of X and postfixing the conjugate reciprocal of any undegenerate simple minor square matrix of the post-factor of X	309-357

§§		PAGES
84.	Cancellation of matrix factors in a matrix equation: a pre-factor or post-factor common to both sides in a matrix equation can be cancelled when and only when its rank is equal to its passivity	357–358
85.	Cancellation of matrix factors in a matrix identity	358–363

CHAPTER XI.

SOLUTION OF ANY SYSTEM OF LINEAR ALGEBRAIC EQUATIONS.

86.	Connections between linear functions and linear equations	364–365
87–88.	Solution of any system of linear equations regarded as equivalent to a single matrix equation. Special case of n unconnected equations in n variables	365–379
89.	Solution of a system of homogeneous linear equations regarded as equivalent to a single matrix equation. Special case of $n-1$ unconnected homogeneous equations in n variables	380–386
90–91.	Unconnected solutions of a system of homogeneous linear equations; symmetrical form of the general solution. Determination of all possible complete sets of unconnected solutions; extravagant and non-extravagant systems; extravagant and non-extravagant solutions	386–394
92–93.	Unconnected solutions of any system of linear equations; symmetrical forms of the general solution. Determination of all possible complete sets of unconnected solutions	395–404
94–95.	Mutually orthogonal solutions of any system of linear equations. Determination of complete sets of mutually orthogonal real unit solutions when the system is real	404–412
96.	Theorems relating to the connections between linear equations	412–415
97.	Functional dependences between functions of several independent variables	415–417
	INDEX	419