# CHAPTER I.

## INTRODUCTION OF RECTANGULAR MATRICES AND DETERMINOIDS.

[This Chapter contains an introductory account of rectangular matrices and determinoids and a description of various abbreviated notations which will be used in connection with them.]

### § 1. Rectangular Matrices.

1. *Definition.*

If $m$ and $n$ are positive integers, an aggregate of $mn$ quantities or *elements* arranged in $m$ horizontal rows and $n$ vertical rows will be called a rectangular matrix. For any such matrix $A$ we may use the notation

$$A = [a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{.........................(1).}$$

The numbers $m$ and $n$ will be called the *orders* of the matrix, $m$ being the *horizontal order* and $n$ the *vertical order*. The smaller of these two numbers (or either of them if they are equal) will be called the effective order or the *efficiency* of the matrix.

If $m$ and $n$ are unequal, either a vertical row contains more elements than a horizontal row, or a horizontal row contains more elements than a vertical row. Those rows, horizontal or vertical, which contain the greater number of elements will be called *long rows*, and the other rows will be called *short rows*. The number of long rows is equal to the efficiency of the matrix.

In the special case in which $m$ and $n$ are equal the matrix will be called a *square matrix*. A square matrix has only one order, which is also its efficiency, and in it either set of parallel rows may be regarded as long rows.

By the *leading element* of the matrix $A$ is meant the element $a_{11}$ in the top left-hand corner. The elements $a_{11}, a_{22}, \dots a_{ii}, \dots$ are said to form the *leading line* of the matrix. In the case of a square matrix the leading line becomes the *leading diagonal*.

The vertical row on the extreme left in $A$ will be called the *leading vertical row* of $A$, and any set of vertical rows lying to the left of all other vertical rows will be said to occupy a *leading position* in $A$. So the topmost horizontal row will be called the *leading horizontal row*, and any set of horizontal rows lying above all other horizontal rows occupy a *leading position* in $A$.

It will be assumed that the elements of a matrix are scalar numbers or letters denoting scalar numbers. The equality of matrices and the addition, subtraction and multiplication of matrices are defined in Chapter VI. In the first five chapters, which deal with properties of a single given matrix, these latter definitions will not be required.

*Ex.* i. In the matrix

$$[a]_3^5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

the long rows are horizontal and the short rows vertical. The horizontal and vertical orders are 3 and 5 respectively. The efficiency is 3. The leading line is that occupied by the elements $a_{11}$, $a_{22}$, $a_{33}$.

*Ex.* ii. In the matrix

$$[abc]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

the long rows are vertical and the short rows horizontal. The horizontal and vertical orders are 4 and 3 respectively. The efficiency is 3. The leading line is that occupied by the elements $a_1$, $b_2$, $c_3$.

## 2. *Derived Matrices.*

A matrix formed from $A$ by re-arranging the horizontal rows amongst themselves in any manner and also re-arranging the vertical rows amongst themselves in any manner will be called a *deranged matrix* of $A$ or a *derangement* of $A$.

Any matrix $A'$ formed from $A$ by simply striking out some of its horizontal and vertical rows and leaving the retained horizontal and vertical rows in the same relative orders as in $A$ will be called a *corranged minor matrix* of $A$. In particular by striking out all horizontal rows below a certain horizontal row and all vertical rows to the right of a certain vertical row we form a corranged minor matrix occupying a *leading position* in $A$.

Any matrix $A''$ formed from a corranged minor matrix $A'$ by re-arranging the retained horizontal rows amongst themselves in any manner and also re-arranging the retained vertical rows amongst themselves in any manner will be called a *deranged minor matrix* of $A$.

A minor matrix formed from $A$ by striking out rows of one kind only, horizontal or vertical, is called a *simple minor matrix*.

The corranged and deranged minor matrices of $A$ and the derangements of $A$ together constitute all the *derived matrices* of $A$.

Every minor matrix of $A$, whether corranged or deranged, has at least one of its orders less than the corresponding order of $A$, and the elements of any one of its horizontal or vertical rows all lie in a parallel row of $A$. Thus the most general form of a minor matrix of $A$ is

$$A'' = [a_{pq}]_{\mu}^{\nu} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} \dots a_{p_1q_\nu} \\ a_{p_2q_1} & a_{p_2q_2} \dots a_{p_2q_\nu} \\ \dots\dots & \dots\dots\dots \\ a_{p_\mu q_1} & a_{p_\mu q_2} \dots a_{p_\mu q_\nu} \end{bmatrix} \quad \dots\dots\dots\dots\dots(2),$$

where $p_1, p_2, \dots p_\mu$ is some arrangement of $\mu$ of the numbers $1, 2, \dots m$, and $q_1, q_2, \dots q_\nu$ is some arrangement of $\nu$ of the numbers $1, 2, \dots n$.

A matrix will be said to *contain* each of its minor matrices.

*Ex.* iii. With respect to the matrix

$$[abcd]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

the following three matrices are minor matrices:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \end{bmatrix}, \quad \begin{bmatrix} c_2 & a_2 & d_2 \\ c_3 & a_3 & d_3 \\ c_1 & a_1 & d_1 \end{bmatrix}.$$

The first two of these are corranged minor matrices, and the first occupies a leading position. The third is a deranged simple minor matrix.

3. *Derived products.*

If as large a number as possible of elements $\alpha, \beta, \gamma, \delta, \dots$ are selected from the matrix $A$ in such a manner that no two of the selected elements occur in the same horizontal row and no two in the same vertical row, their product $\alpha\beta\gamma\delta\dots$ will be called a *complete derived product* belonging to the matrix. The number of elements in each complete derived product is equal to the efficiency of the matrix; and the total number of such products, when the order of arrangement of the factors is disregarded, is

$$n(n-1)(n-2)\dots(n-m+1) \quad \text{or} \quad m(m-1)(m-2)\dots(m-n+1)$$

according as $n$ is greater or less than $m$. To form any particular complete derived product each long row contributes just one factor, but (except in the case of a square matrix) there are short rows which do not contribute any factor.

More generally we will define a *derived product of order r* belonging to the matrix $A$ to be a product $\alpha\beta\gamma\delta\ldots$ formed with $r$ elements $\alpha, \beta, \gamma, \delta, \ldots$ selected from the matrix $A$ in such a manner that no two of the elements occur in the same horizontal row and no two in the same vertical row. Then $r$ cannot be greater than the efficiency of the matrix, and the derived product is *complete* or *incomplete* according as $r$ is equal to or less than the efficiency. The most general form of a derived product $P$ of order $r$ belonging to the matrix $A$ is

$$P = a_{p_1 q_1}\, a_{p_2 q_2} \ldots a_{p_r q_r} \qquad \text{.........................(3)},$$

where $p_1, p_2, \ldots p_r$ is any arrangement of $r$ of the numbers $1, 2, \ldots n$,

and $q_1, q_2, \ldots q_r$ is any arrangement of $r$ of the numbers $1, 2, \ldots m$.

The total number of derived products of order $r$ belonging to the matrix $A$ is

$$r!\binom{m}{r}\binom{n}{r} = \frac{m(m-1)(m-2)\ldots(m-r+1).\,n(n-1)(n-2)\ldots(n-r+1)}{r!}.$$

*Ex.* iv. In the matrix

$$[abcde]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}$$

$c_1 d_2 b_3 e_4$ is a complete derived product;

$d_4 b_2 a_3$ is an (incomplete) derived product of order 3.

4. *Steps.*

The passage from any element of the matrix to an adjacent element in the same horizontal row will be called a *forward* or *backward horizontal step* according as the second element lies to the right or to the left of the first. Similarly the passage from any element to an adjacent element in the same vertical row will be called a *forward* or *backward vertical step* according as the second element lies below or above the first. When the word *step* is used without qualification it will be understood to mean a forward step.

*Ex.* v. In the matrix

$$[abcde]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}$$

the passage from $b_3$ to $c_3$ is a (forward) horizontal step, and the passage from $d_2$ to $d_3$ is a (forward) vertical step. The passage from $a_1$ to $c_4$ requires two horizontal steps and three vertical steps.

5. *Conjugate matrices.*

Two matrices $A$ and $A'$ are said to be *conjugate* to one another when the horizontal and vertical rows of the one taken in order are respectively the vertical and horizontal rows of the other taken in order. The conjugate matrix $A'$ of the matrix $A$ defined by equation (1) above will be denoted by

$$A' = \overline{\underline{a}}{}^{m}_{n} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots(4).$$

A *self-conjugate matrix* is a square matrix which is symmetrical with respect to its leading diagonal. Thus the matrix $[a]^{m}_{m}$ is self-conjugate if $a_{ij} = a_{ji}$, where $i$ and $j$ are any two of the numbers $1, 2, \dots m$.

*Ex.* vi. The two matrices

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}$$

are conjugate to one another.

*Ex.* vii. The matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a self-conjugate matrix.

## § 2. Abbreviated notations for a matrix.

In all the following defining formulae $A$ and $A'$ are a pair of mutually conjugate matrices.

1. *Standard double-suffix notation.*

$$\left.\begin{aligned} A = [a]^{n}_{m} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \\ A' = \overline{\underline{a}}{}^{m}_{n} &= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots(A).$$

Here $[a]_m^n$, $\overline{\underline{a}}{}_n^m$ are to be regarded as abbreviations for the matrices following them on the right. This is the notation which will most commonly be employed in proving general theorems relating to matrices. The general symbol for an element of either matrix is $a_{xy}$, and the various individual elements are obtained from this by giving to $x$ the values $1, 2, \dots m$ and to $y$ the values $1, 2, \dots n$. With respect to the matrix $[a]_m^n$, the suffixes $x$ and $y$ will be called the *vertical* and *horizontal suffixes* respectively of the element $a_{xy}$, since they indicate its vertical and horizontal positions relative to the leading element $a_{11}$. With respect to the matrix $\overline{\underline{a}}{}_n^m$ these terms must be reversed.

*Ex.* i. $[a]_2^3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $\overline{\underline{a}}{}_3^2 = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$.

2. *More general double-suffix notations.*

In order to represent the minor matrices of a matrix $[a]_\mu^\nu$, other more elaborate notations are necessary. All such minors can be represented by symbols $[a_{pq}]_m^n$, $\overline{\underline{a_{pq}}}{}_n^m$ defined as follows:

$$\left.\begin{aligned} A = [a_{pq}]_m^n &= \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \cdots & a_{p_1q_n} \\ a_{p_2q_1} & a_{p_2q_2} & \cdots & a_{p_2q_n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_mq_1} & a_{p_mq_2} & \cdots & a_{p_mq_n} \end{bmatrix} \\ A' = \overline{\underline{a_{pq}}}{}_n^m &= \begin{bmatrix} a_{p_1q_1} & a_{p_2q_1} & \cdots & a_{p_mq_1} \\ a_{p_1q_2} & a_{p_2q_2} & \cdots & a_{p_mq_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_1q_n} & a_{p_2q_n} & \cdots & a_{p_mq_n} \end{bmatrix} \end{aligned}\right\} \dots\dots\dots\dots (B).$$

Here the general symbol for an element of either matrix is $a_{p_xq_y}$, and the various individual elements are obtained from this by giving to $x$ the values $1, 2, \dots m$ and to $y$ the values $1, 2, \dots n$. Keeping $x$ constant we obtain all the elements of the $x$th horizontal row of $A$, which is the $x$th vertical row of $A'$. So keeping $y$ constant we obtain all the elements of the $y$th vertical row of $A$, which is the $y$th horizontal row of $A'$. The suffixes $p_x$, $q_y$ will be called respectively the vertical and the horizontal suffix of the element $a_{p_xq_y}$ in the matrix $A$, and they will be called respectively the horizontal and the vertical suffix of that element in the matrix $A'$.

We will also define symbols $[a_{p1}]_m^n$, $\underline{\overline{a_{p1}}}_n^m$ by the equations

$$\left.\begin{aligned} A = [a_{p1}]_m^n &= \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & \dots & a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} & \dots & a_{p_2 n} \\ \dots & \dots & \dots & \dots \\ a_{p_m 1} & a_{p_m 2} & \dots & a_{p_m n} \end{bmatrix} \\ A' = \underline{\overline{a_{p1}}}_n^m &= \begin{bmatrix} a_{p_1 1} & a_{p_2 1} & \dots & a_{p_m 1} \\ a_{p_1 2} & a_{p_2 2} & \dots & a_{p_m 2} \\ \dots & \dots & \dots & \dots \\ a_{p_1 n} & a_{p_2 n} & \dots & a_{p_m n} \end{bmatrix} \end{aligned}\right\} \quad \text{.................(C).}$$

Here the general symbol for an element of either matrix is $a_{p_x y}$, and the various individual elements are obtained from this by giving to $x$ the values $1, 2, \dots m$, and to $y$ the values $1, 2, \dots n$. This notation can be conveniently employed to represent the various corranged simple minor matrices obtained from a matrix $[a]_\mu^n$ by striking out vertical rows.

Similarly we will define symbols $[a_{1q}]_m^n$, $\underline{\overline{a_{1q}}}_n^m$ by the equations

$$\left.\begin{aligned} A = [a_{1q}]_m^n &= \begin{bmatrix} a_{1q_1} & a_{1q_2} & \dots & a_{1q_n} \\ a_{2q_1} & a_{2q_2} & \dots & a_{2q_n} \\ \dots & \dots & \dots & \dots \\ a_{mq_1} & a_{mq_2} & \dots & a_{mq_n} \end{bmatrix} \\ A' = \underline{\overline{a_{1q}}}_n^m &= \begin{bmatrix} a_{1q_1} & a_{2q_1} & \dots & a_{mq_1} \\ a_{1q_2} & a_{2q_2} & \dots & a_{mq_2} \\ \dots & \dots & \dots & \dots \\ a_{1q_n} & a_{2q_n} & \dots & a_{mq_n} \end{bmatrix} \end{aligned}\right\} \quad \text{.................(D).}$$

Here the general symbol for an element of either matrix is $a_{xq_y}$ and the various individual elements are obtained from this by giving to $x$ the values $1, 2, \dots m$ and to $y$ the values $1, 2, \dots n$. This is a convenient notation for the corranged simple minor matrices of a matrix $[a]_m^\nu$ obtained by striking out horizontal rows.

*Ex.* ii. $[a_{pq}]_2^3 = \begin{bmatrix} a_{p_1 q_1} & a_{p_1 q_2} & a_{p_1 q_3} \\ a_{p_2 q_1} & a_{p_2 q_2} & a_{p_2 q_3} \end{bmatrix}$, $\quad \underline{\overline{a_{pq}}}_3^2 = \begin{bmatrix} a_{p_1 q_1} & a_{p_2 q_1} \\ a_{p_1 q_2} & a_{p_2 q_2} \\ a_{p_1 q_3} & a_{p_2 q_3} \end{bmatrix}$.

*Ex.* iii. $[a_{p1}]_2^3 = \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & a_{p_1 3} \\ a_{p_2 1} & a_{p_2 2} & a_{p_2 3} \end{bmatrix}$, $\quad \underline{\overline{a_{p1}}}_3^2 = \begin{bmatrix} a_{p_1 1} & a_{p_2 1} \\ a_{p_1 2} & a_{p_2 2} \\ a_{p_1 3} & a_{p_2 3} \end{bmatrix}$.

3. *Most general double-suffix notation.*

Other symbols $\begin{bmatrix} uv\ldots w \\ a \\ pq\ldots r \end{bmatrix}$, $\overline{\underline{\begin{matrix} pq\ldots r \\ a \\ uv\ldots w \end{matrix}}}$ will be defined by the equations

$$\left.\begin{aligned} A &= \begin{bmatrix} uv\ldots w \\ a \\ pq\ldots r \end{bmatrix} = \begin{bmatrix} a_{pu} & a_{pv} & \ldots & a_{pw} \\ a_{qu} & a_{qv} & \ldots & a_{qw} \\ \ldots & \ldots & \ldots & \ldots \\ a_{ru} & a_{rv} & \ldots & a_{rw} \end{bmatrix} \\ A' &= \overline{\underline{\begin{matrix} pq\ldots r \\ a \\ uv\ldots w \end{matrix}}} = \begin{bmatrix} a_{pu} & a_{qu} & \ldots & a_{ru} \\ a_{pv} & a_{qv} & \ldots & a_{rv} \\ \ldots & \ldots & \ldots & \ldots \\ a_{pw} & a_{qw} & \ldots & a_{rw} \end{bmatrix} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(E).$$

All the matrices occurring in formulae (A), (B), (C), (D) can be represented in this form.

*Ex.* iv. $[a]_m^n = \begin{bmatrix} 12\ldots n \\ a \\ 12\ldots m \end{bmatrix}$, $[a_{pq}]_m^n = \begin{bmatrix} q_1 q_2 \ldots q_n \\ a \\ p_1 p_2 \ldots p_m \end{bmatrix}$, $[a_{1q}]_m^n = \begin{bmatrix} q_1 q_2 \ldots q_n \\ a \\ 12\ldots m \end{bmatrix}$.

4. *Double-suffix notation for augmented matrices.*

Symbols $[a, b]_m^{n,r}$, $\begin{bmatrix} a \\ b \end{bmatrix}_{m,r}^n$ will be defined by

$$[a, b]_m^{n,r} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_{11} & b_{12} & \ldots & b_{1r} \\ a_{21} & a_{22} & \ldots & a_{2n} & b_{21} & b_{22} & \ldots & b_{2r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_{m1} & b_{m2} & \ldots & b_{mr} \end{bmatrix}, \quad \begin{bmatrix} a \\ b \end{bmatrix}_{m,r}^n = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \\ b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix}$$

$$\ldots\ldots\ldots\ldots(F).$$

These matrices will be called *augmented matrices* of $[a]_m^n$. The first is formed by placing the vertical rows of $[b]_m^r$ to the right of the vertical rows of $[a]_m^n$. The second is formed by placing the horizontal rows of $[b]_r^n$ below the horizontal rows of $[a]_m^n$.

The conjugate matrices of $[a, b]_m^{n,r}$, $\begin{bmatrix} a \\ b \end{bmatrix}_{m,r}^n$ will be denoted by

$$\overline{\underline{\begin{matrix} a \\ b \end{matrix}}}{}_{n,r}^{m} \quad \text{and} \quad \overline{\underline{a, b}}{}_n^{m,r}.$$

Similarly we will write

$$[a_{pq}, b_{pu}]_m^{n,r} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \cdots & a_{p_1q_n} & b_{p_1u_1} & b_{p_1u_2} & \cdots & b_{p_1u_r} \\ a_{p_2q_1} & a_{p_2q_2} & \cdots & a_{p_2q_n} & b_{p_2u_1} & b_{p_2u_2} & \cdots & b_{p_2u_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p_mq_1} & a_{p_mq_2} & \cdots & a_{p_mq_n} & b_{p_mu_1} & b_{p_mu_2} & \cdots & b_{p_mu_r} \end{bmatrix} \quad \text{......(G)},$$

$$\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,r}^{n} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \cdots & a_{p_1q_n} \\ a_{p_2q_1} & a_{p_2q_2} & \cdots & a_{p_2q_n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_mq_1} & a_{p_mq_2} & \cdots & a_{p_mq_n} \\ b_{v_1q_1} & b_{v_1q_2} & \cdots & b_{v_1q_n} \\ b_{v_2q_1} & b_{v_2q_2} & \cdots & b_{v_2q_n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{v_rq_1} & b_{v_rq_2} & \cdots & b_{v_rq_n} \end{bmatrix} \quad \text{....................(H)}.$$

Also $[a_{1q}, b_{1u}]_m^{n,r}$ will denote the matrix formed by placing the vertical rows of $[b_{1u}]_m^r$ to the right of the vertical rows of $[a_{1q}]_m^n$, and $\begin{bmatrix} a_{p1} \\ b_{v1} \end{bmatrix}_{m,r}^{n}$ will denote the matrix formed by placing the horizontal rows of $[b_{v1}]_r^n$ below the horizontal rows of $[a_{p1}]_m^n$. Thus $[a_{1q}, b_{1u}]_m^{n,r}$ is formed from $[a_{pq}, b_{pu}]_m^{n,r}$ by putting $p_1, p_2, \ldots p_m = 1, 2, \ldots m$, and $\begin{bmatrix} a_{p1} \\ b_{v1} \end{bmatrix}_{m,r}^{n}$ is formed from $\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,r}^{n}$ by putting $q_1, q_2, \ldots q_n = 1, 2, \ldots n$.

The conjugates of $[a_{pq}, b_{pu}]_m^{n,r}$, $\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,r}^{n}$ may be denoted by

$$\left\ulcorner \begin{matrix} a_{pq} \\ b_{pu} \end{matrix} \right\lrcorner_{n,r}^{m} \quad \text{and} \quad \left\ulcorner \; a_{pq}, \; b_{vq} \; \right\lrcorner_{n}^{m,r}.$$

*Ex.* v. $[a, b]_3^{2,3} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & b_{31} & b_{32} & b_{33} \end{bmatrix}$.

*Ex.* vi. $\begin{bmatrix} a_{p1} \\ b_{q1} \end{bmatrix}_{2,2}^{3} = \begin{bmatrix} a_{p_11} & a_{p_12} & a_{p_13} \\ a_{p_21} & a_{p_22} & a_{p_23} \\ b_{q_11} & b_{q_12} & b_{q_13} \\ b_{q_21} & b_{q_22} & b_{q_23} \end{bmatrix}$.

5. *Standard single-suffix notation.*

$$\left.\begin{aligned} A = [ab \dots k]_{12\dots m} &= \begin{bmatrix} a_1 & b_1 & \dots & k_1 \\ a_2 & b_2 & \dots & k_2 \\ \dots & \dots & \dots & \dots \\ a_m & b_m & \dots & k_m \end{bmatrix} \\ A' = \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{12\dots m} &= \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \\ \dots & \dots & \dots & \dots \\ k_1 & k_2 & \dots & k_m \end{bmatrix} \end{aligned}\right\} \dots\dots\dots\dots(\text{I}).$$

This is a convenient notation when we are considering a matrix whose orders are given small numbers or when we are considering the simple minor matrices of a given matrix. When no ambiguity will be thereby caused, the suffixes in the abbreviated symbols can be omitted. In particular they will often be omitted when the matrix is a square matrix.

*Ex.* vii. The matrix

$$[abc]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

will often be denoted simply by $[abc]$.

*Ex.* viii. The corranged simple minors of the matrix

$$[abcd]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

obtained by striking out two vertical rows are

$$[ab]_{123},\quad [ac]_{123},\quad [ad]_{123},\quad [bc]_{123},\quad [bd]_{123},\quad [cd]_{123}.$$

These will often be denoted simply by

$$[ab],\ [ac],\ [ad],\ [bc],\ [bd],\ [cd].$$

6. *Most general single-suffix notation.*

In order to represent all the minors, or even all the simple minors, of a matrix of the form $[ab \dots k]_{12\dots m}$, a more general notation is required. Accordingly we shall define symbols

$$[ab \dots k]_{\alpha\beta\dots\kappa},\quad \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{\alpha\beta\dots\kappa}$$