MATHEMATICAL MODELLING IN ONE DIMENSION

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An Introduction via Difference and Differential Equations

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107654686

First published 2013

Printed and bound in the United Kingdom by the MPG Books Group

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-65468-6 Paperback

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Preface

Engineers, natural scientists and, increasingly, researchers and practitioners working in economics and other social sciences, use mathematical modelling to solve problems arising in their disciplines. There are at least two identifiable kinds of mathematical modelling. One involves translating the rules of nature or society into mathematical formulae, applying mathematical methods to analyse them and then trying to understand the implications of the obtained results for the original disciplines. The other kind is to use mathematical reasoning to solve practical industrial or engineering problems without necessarily building a mathematical theory for them.

This book is predominantly concerned with the first kind of modelling: that is, with the analysis and interpretation of models of phenomena and processes occurring in the real world. It is important to understand, however, that models only give simplified descriptions of real-life problems but, nevertheless, they can be expressed in terms of mathematical equations and thus can be solved in one way or another.

Mathematical modelling is a difficult subject to teach but it is what applied mathematics is all about. The difficulty is that there are no set rules and the understanding of the 'right' way to model can be only reached by familiarity with a number of examples. Therefore in this book we shall discuss a wide range of mathematical models referring to real life phenomena and introduce basic techniques for solving and interpreting the solutions of the resulting equations. It is, however, fair to emphasize that this is not a conventional textbook on mathematical modelling. We do not spend too much time on building the models but a central role is played by difference and differential equations and their analysis. However, within the space limitations, we try to describe the origin of the models and interpret the results of analysing them.

Nevertheless, let us briefly describe the full process of mathematical modelling. First, there must be a phenomenon of interest that one wants to describe or, more importantly, explain and make predictions about. Observation of this phenomenon allows one to make hypotheses about those quantities that are most relevant to the problem and what the relations between them are so that one can devise a hypothetical mechanism that describes the phenomenon. At this stage one has to decide how to quantify, or assign numbers to, the observations, e.g. whether the problem is to be set in absolute space-time or in relativistic setting, or whether time should be continuous or discrete, etc. The choice is not always obvious or unique but one needs to decide on a particular approach before one begins to build a model. The purpose of building the model is to formulate a description of the mechanism driving the phenomenon of interest in quantitative terms: that is, as mathematical equations which can be mathematically analysed. After that, it is necessary to interpret the solution, or any other information extracted from the equations, as statements about the original problem so that they can be tested against observations. Ideally, the model also leads to predictions which, if verified, serve as a further validation of the model. It is important to realize that modelling is usually an iterative procedure as it is very difficult to achieve a proper balance between the simplicity and meaningfulness of the model. Often the model turns out to be too complicated to yield itself to analysis or it is over-simplified so that there is insufficient agreement between actual experiment and the results predicted from the model. In both these cases one has to return to the first step of the modelling process to try to remedy the problem.

This first step in modelling is the most creative but also the most difficult, often involving a concerted effort of specialists in many diverse fields. Hence, as we said earlier, though we describe a number of models in detail, starting from first principles, the main emphasis of the course is on the later stages of the modelling process; that is, on analysing and solving the equations, interpreting their solutions in the language of the original problem and reflecting on whether the answers seem reasonable.

In most cases discussed here the model is a representation of a process; that is, it describes a change in the states of some system in time. This description could be discrete and continuous. The former corresponds to the situation in which we observe a system at regular finite time intervals, say, every second or every year, and relate the observed state of the system to the states at the previous instants. Such a system can be modelled by difference equations. In the continuous cases we treat time as a continuum allowing for observations of the system at any time. In such a case the model can express relations between the rates of change of various quantities rather than between the states at various times and, since the rates of change are given by derivatives, the model is represented by differential equations.

Furthermore, models can describe the evolution of a given system either in a non-interacting environment, such as a population of bacteria in a Petri dish, or else engaged in interactions with other systems. In the first case the model consists of a single equation and we say that the model is one-dimensional, while in the second case we have to deal with several (sometimes infinitely many) equations describing the interactions; then the model is said to be multi-dimensional.

In this book we only discuss one-dimensional models. Multi-dimensional models are planned to be the subject of a companion volume.

This book has been inspired and heavily draws on several excellent textbooks such as (Braun, 1983; Elaydi, 2005; Friedman and Littman, 1994; Glendinning, 1994; Strogatz, 1994), to mention but a few. However, I believe that the presented blend of discrete and continuous models and the combination of a detailed description of the modelling process with mathematical analysis of the resulting equations makes it different from any of them. I hope that the readers will find it fills a gap in the existing literature.

This book is based on lectures given, at various levels and for various courses, at the University of KwaZulu-Natal in Durban, at the African Institute of Mathematical Sciences in Muizenberg, South Africa, and at the Technical University of Łódź, Poland. My thanks go to several generations of students on whom I tested continually changing ideas of the course. Working with students allowed me to clarify many presentations and correct numerous mistakes. Finally, I am very grateful to our School Secretary, Dale Haslop, and my PhD student, Eddy Kimba Phongi, who, by combing through the final version of the book, helped to substantially reduce the number of remaining errors and thus to make the book more readable.

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October 2012