Representations of Lie Algebras

This bold and refreshing approach to Lie algebras assumes only modest prerequisites (linear algebra up to the Jordan canonical form and a basic familiarity with groups and rings), yet it reaches a major result in representation theory: the highest-weight classification of irreducible modules of the general linear Lie algebra. The author's exposition is focused on this goal rather than on aiming at the widest generality, and emphasis is placed on explicit calculations with bases and matrices. The book begins with a motivating chapter explaining the context and relevance of Lie algebras and their representations and concludes with a guide to further reading. Numerous examples and exercises with full solutions are included.

Based on the author's own introductory course on Lie algebras, this book has been thoroughly road-tested by advanced undergraduate and beginning graduate students and is also suited to individual readers wanting an introduction to this important area of mathematics.

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Representations of Lie Algebras

An Introduction Through $\mathfrak{gl}_n$

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Preface

The aim of this book

Why another introduction to Lie algebras? The subject of this book is one of the areas of algebra that has been most written about. The basic theory was unearthed more than a century ago and has been polished in a long chain of textbooks to a sheen of classical perfection. Experts' shelves are graced by the three volumes of Bourbaki [1]; for students with the right background and motivation to learn from them, the expositions in the books by Humphreys [10], Fulton and Harris [6], and Carter [2] could hardly be bettered; and there is a recent undergraduate-level introduction by Erdmann and Wildon [4]. So where is the need for this book?

The answer comes from my own experience in teaching courses on Lie algebras to Australian honours-level undergraduates (see the Acknowledgements section). Such courses typically consist of 24 one-hour lectures. At my own university the algebraic background knowledge of the students would be: linear algebra up to the Jordan canonical form, the basic theory of groups and rings, the rudiments of group representation theory, and a little multilinear algebra in the context of differential forms. From that starting point, I have found it difficult to reach any peak of the theory by following the conventional route. My definition of a peak includes the classification of simple Lie algebras, the highest-weight classification of their modules, and the combinatorics of characters, tensor products, and crystal bases; by 'the conventional route' I mean the path signposted by the theorems of Engel and Lie (about solvability), Cartan (about the Killing form), Weyl (about complete reducibility), and Serre, as in the book by Humphreys [10]. Following that path without skipping proofs always seemed to require more than 24 lectures.

The solution adopted in this book is drastic. I have abandoned the wider class of simple Lie algebras, focusing instead on the general linear Lie algebra $\mathfrak{gl}_n$, which is almost, but not quite, simple. I have jettisoned all five of the aforementioned theorems, in favour of arguments specific to $\mathfrak{gl}_n$, especially the use of explicit Casimir operators. Although these omissions may shock the experts, I have found this to be an approach that is more accessible and yet still reaches one peak: the classification of $\mathfrak{gl}_n$-modules by their highest weights.

I have started the journey with a motivatory chapter, which gives some explanation of why algebraists care about this classification and also introduces some necessary multilinear algebra. Chapters 2 to 4 cover the basic definitions of Lie algebras, homomorphisms and isomorphisms, subalgebras, ideals, quotients, modules, irreducibility and complete reducibility. In a lecture course, the material in these first four chapters would typically take about 12 hours; so the elegant $\mathfrak{sl}_2$ theory in Chapter 5 is reached relatively early. Then in Chapter 6 I return to the theory of modules, covering tensor products, bilinear forms, Schur's lemma, and Casimir operators.

In Chapter 7 these tools are used to develop the highest-weight theory. My hope is that students who reach the end of Chapter 7 will be inspired to progress to more comprehensive books, and Chapter 8 is intended as a map of what lies ahead.

Acknowledgements

This book began life as a set of lecture notes for my Lie algebras course in the 2004 Australian Mathematical Sciences Institute (AMSI) Summer School. It was extensively revised over the next seven years, as I taught the subject again for the summer school and as an honours course at the University of Sydney. Most of the exercises were originally assignment or exam questions.

I would like to thank AMSI for the initial opportunity to teach this beautiful subject, and the students in all those classes for their feedback. I would also like to thank Pramod Achar, Wai Ling Yee, Cheryl Praeger, and the anonymous reviewers for their valuable suggestions and encouraging comments.

Notational conventions

To simplify matters, we make a standing convention:

All vector spaces are over $\mathbb{C}$ and finite-dimensional.

The finite-dimensionality assumption allows the explicit calculations with bases and matrices that are a feature of the book. The $n \times n$ identity matrix is written 1_n, and the identity transformation of a vector space V is written 1_V. The elements of the vector space $\mathbb{C}^n$ are always thought of as column vectors; linear transformations of this particular vector space are tacitly identified with $n \times n$ matrices (multiplying on the left of the vectors). The bases of vector spaces are considered to be ordered sets and hence are written without set braces. The span of the elements $v_1, \ldots, v_k$ is written $\mathbb{C}\{v_1, \ldots, v_k\}$. The term 'subspace' always means 'sub-vector-space'. If W and W' are subspaces of a larger vector space V then $W \oplus W'$ denotes their sum, and it is implied that $W \cap W' = \{0\}$ (an 'internal direct sum'); if W and W' are not subspaces of a larger vector space V then $W \oplus W'$ means the 'external direct sum' $\{(w, w') \mid w \in W, w' \in W'\}$. The same principles apply to direct sums with more than two summands.

On its rare appearances, the square root of -1 is written $\mathbf{i}$ to distinguish it from the italic letter i, which is widely used for other purposes. The group of nonzero complex numbers is written $\mathbb{C}^\times$. The set of nonnegative integers is written $\mathbb{N}$. Other notation will be explained as it is needed.