CHAPTER I

ELEMENTARY THEORY OF HARMONIC MOTION

1. **Introduction.** In experimental physics, it very often happens that the system, whose motions are observed, performs oscillations and it is not difficult to give practical reasons why this type of motion occurs so much more frequently than any other type.

To begin with, it is but seldom we can observe a body moving uniformly along a straight line. The slow fall with a limiting velocity of a small sphere through a viscous liquid is one instance, but it would be difficult to name many more.

When a body is falling freely or is under the action of a "diluted" gravitational action, as when a sphere rolls down an inclined plane, or two bodies move in an Atwood's machine, the motion suffers uniform acceleration in a straight line, but it is perhaps only by means of gravity that uniform acceleration in a straight line can be obtained. Uniformly accelerated motions have the practical disadvantage that the interval of time to be measured is short, when the motion is limited to a few metres, as it is in an ordinary room, unless the acceleration is very slow, in which case the effects of disturbing forces may rival the effects we wish to study.

When a body is compelled to turn about a fixed axis, the space required for the movement is comparatively small, so that one of the objections to uniform or uniformly accelerated motion in a straight line does not apply. In the case of uniform angular velocity, a supply of power is required to maintain the motion, as when the disc of a siren is driven by an electromotor. In such a case we are, however, generally but little concerned with the dynamics of the apparatus, our attention being confined to producing a uniform rotation somehow.

To produce uniform angular acceleration in the absence of friction, a couple of constant magnitude would be necessary, but such a couple it would be very difficult, if not impossible, to produce.

The reader will thus perceive that he is likely to meet with very few instances of uniform or uniformly accelerated motion either along a straight line or round a fixed axis.

When a vibratory motion is substituted for one in which the movement is always in one direction, a great advantage is at once gained. For now, even in the case of rectilinear motion, only a comparatively small space is required; and in both rectilinear and angular motions, although the time of one vibration may be small, it can be found with considerable accuracy by observing the time occupied by a large number of vibrations, as can be done if the vibrations die only slowly away. In most cases, further, the time of vibration is practically independent of the amplitude of vibration, so long as the amplitude is "small."

2. Harmonic motion. On a circle with O (Fig. 1) for its centre, take a point P and draw a perpendicular PN from P upon any diameter AOA'. Then, if P move round the circle with uniform angular velocity, the point N will move backwards and forwards along AOA' in a definite manner, and the motion of N is called harmonic. The length OA is called the amplitude of the oscillation, and the time occupied by N in going from A to A' and back to A is called the time of a complete vibration, or the periodic time.

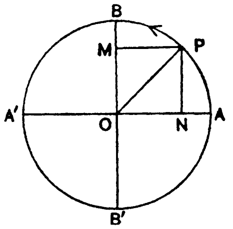

Fig. 1.

Since N is the foot of the perpendicular PN, the velocity and acceleration of N along AOA' are equal to the components, parallel to AOA', of the velocity and acceleration of P.

Let OP revolve in the counter-clockwise direction, let ω radians per second be the angular velocity of OP, let the radius of the circle be r cm., and let $ON = x$ cm., OA being the positive direction of x. Then, if t seconds be the time since P was last at A, the angle POA is ωt radians. Hence we have

$$x = ON = r \cos \omega t, \quad \text{.....................(1)}$$

and thus x is proportional to $\cos \omega t$. If PM be drawn perpendicular to OB, where OB is perpendicular to OA, the motion of M will be harmonic also and we shall have

$$y = OM = r \sin \omega t. \quad \text{.......................(2)}$$

The functions $\cos \omega t$ and $\sin \omega t$ occur in the theory of the vibrations of stretched strings, and it is from the connexion of such strings with the musical scale that the use of the adjective *harmonic* has been extended to the motion of a point whose displacement is proportional to $\cos \omega t$ or $\sin \omega t$.

3. The velocity. The length of arc passed over by P in one second is r times the angle turned through by OP in one second, and hence, if v cm. sec.$^{-1}$ be the velocity of P along the circumference of the circle,

$$v = r\omega. \quad \text{...............................(3)}$$

Since this velocity is perpendicular to OP, its component parallel to OA is $-r\omega \sin POA$, and thus, if u be the velocity of N along OA,

$$u = -r\omega \sin \omega t = -v \sin \omega t. \quad \text{...............(4)}$$

4. The acceleration. Since u is the rate at which x increases with the time, the rate of increase of $r \cos \omega t$ is $-r\omega \sin \omega t$. Writing $\omega t + \frac{1}{2}\pi$ for ωt in this expression, and multiplying by ω, we see that the rate of increase of $r\omega \cos (\omega t + \frac{1}{2}\pi)$ or of $-r\omega \sin \omega t$ is $-r\omega^2 \sin (\omega t + \frac{1}{2}\pi)$ or $-r\omega^2 \cos \omega t$. But the rate of increase of u is the acceleration of N, and hence, if f cm. sec.$^{-2}$ be the acceleration of N in the direction OA,

$$f = -r\omega^2 \cos \omega t = -\omega^2 x. \quad \text{...................(5)}$$

From this equation it will be seen that, when x is positive, f is negative and *vice versa*.

Hence, when a point moves with harmonic motion along a straight line, its acceleration is always directed towards the centre O, and is proportional to its distance from O.

Since, by (3), $\omega = v/r$, we have $r\omega^2 = v^2/r$, and thus

$$f = -\frac{v^2}{r}\cos \omega t. \quad \text{.......................(6)}$$

5. Application of the calculus. The velocity and the acceleration of the moving point N (Fig. 1) can be readily found by the use of the calculus.

Since, by (1), $x = r\cos \omega t$,

we have $u = dx/dt = -r\omega \sin \omega t$

and $f = d^2x/dt^2 = du/dt = -r\omega^2 \cos \omega t$,

so that $f = -\omega^2 x$,

as found in § 4.

6. Acceleration of a point in uniform circular motion. The acceleration of N may also be deduced from the acceleration of a point moving uniformly round a circle.

Let P (Fig. 2) be a point moving round a circle of radius r with uniform velocity v, the angular velocity of the radius OP being ω. When the point is at P, it is moving along the tangent PT with the velocity v, and when it is at P', it is moving along the tangent $P'T'$ with the velocity v. If t be the time of describing PP', the angle POP' is equal to ωt. Now the velocity of the moving point at P' can be resolved into $v \sin \omega t$ parallel to PO, and $v \cos \omega t$ parallel to PT. In the time t, the point has gained the velocity $v \sin \omega t$ parallel to PO, and hence, if a_{av} be its average acceleration parallel to PO,

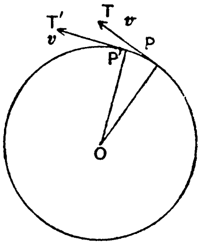

Fig. 2.

$$a_{av} = \frac{v \sin \omega t}{t} = v\omega \frac{\sin \omega t}{\omega t}.$$

As t and ωt approach zero, the average value, during the journey from P to P', of the acceleration of the point parallel to PO, approaches a limiting value, which is its actual acceleration a in the direction from P to O when the point is at P, and thus the acceleration of the point at P is given by the limiting value which a_{av} approaches as t approaches zero. Since $\sin \omega t/\omega t$ approaches the limiting value unity as ωt approaches zero, the acceleration in the direction PO, of the point when at P, is given by

$$\alpha = v\omega. \qquad\qquad (7)$$

We must now see whether the point, as it passes through P, has or has not any acceleration parallel to the tangent PT. If β_{av} be the average acceleration of the point parallel to PT,

$$\beta_{av} = (1/t)(v\cos\omega t - v) = -(2v/t)\sin^2 \tfrac{1}{2}\omega t$$
$$= -v\omega\frac{\sin\frac{1}{2}\omega t}{\frac{1}{2}\omega t}.\sin\tfrac{1}{2}\omega t,$$

and this approaches the limiting value zero when t approaches zero. Thus, when the moving point is at P, it has no acceleration along the tangent at P.

Hence, when a point moves uniformly round a circle, the only acceleration which it has is towards the centre at each instant.

Since $v = r\omega$, the acceleration α can be expressed by any of the three following formulae:—

$$\alpha = v\omega \qquad\qquad (8)$$
$$= v^2/r \qquad\qquad (9)$$
$$= \omega^2 r. \qquad\qquad (10)$$

We can now deduce the acceleration of N (Fig. 1) from that of P. Since the acceleration of P is $\omega^2 r$ or $\omega^2 . PO$ in the direction PO, it follows, from the triangle of accelerations, that the acceleration of P parallel to AO is $\omega^2 . NO$ or $\omega^2 x$ *towards* O. But the acceleration of N is equal to the component of the acceleration of P parallel to AO, and thus f, the acceleration of N in the *positive* direction OA, is given by

$$f = -\omega^2 x. \qquad\qquad (11)$$

7. The periodic time. As the point P goes round the circle, the point N (Fig. 1) oscillates along AOA' and the times of a complete oscillation of N and of a complete revolution of P are equal. Hence, if the time of a complete oscillation of N, or the periodic time of N, be T seconds, the radius OP describes the angle 2π radians in a time T seconds when moving with the angular velocity ω radians per second.

Hence $$T = \frac{2\pi}{\omega}. \qquad\qquad (12)$$

Since ω^2 is equal to the acceleration which the point has towards O when x, the displacement, is unity, this result can be written

$$T = \frac{2\pi}{\sqrt{\text{acceleration for unit displacement}}}. \quad \text{......(13)}$$

If a starting point N_0 be chosen on AA' and if at a given instant N is moving through N_0 in a given direction, it is clear that the interval, which elapses before N is again moving through N_0 in the same direction (for the first time), is independent of the position of N_0.

8. Isochronism. The radius of the auxiliary circle does not appear in the formula for the periodic time and hence T is independent of the amplitude. The extent of the oscillation has therefore no influence upon the time of a complete oscillation. In consequence of this property, which is obviously of great importance, the vibrations are called *isochronous*.

9. Summary of results. The results we have obtained may be restated as follows:—If a point N moving along a straight line have an acceleration μx towards a fixed point O on this line, where x is the distance of N from O, the acceleration when there is unit displacement is μ. Hence, by §§ 7, 8, the point performs isochronous vibrations in the time T, where

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\text{acceleration for unit displacement}}}.$$

If T be observed, the value of μ can be found from the equation

$$\mu = \frac{4\pi^2}{T^2}. \quad \text{..........................(14)}$$

10. Differential equation of harmonic motion. The time of oscillation of a point moving along the axis of x and having the acceleration $-\mu x$ can be readily deduced by the calculus. For the equation of motion

$$\frac{d^2x}{dt^2} = -\mu x \quad \text{..........................(15)}$$

is satisfied by

$$x = r\cos\mu^{\frac{1}{2}}t + s\sin\mu^{\frac{1}{2}}t, \quad \text{..................(16)}$$

where r and s are any constants. If the origin of time be so chosen that, when $\mu^{\frac{1}{2}}t = \frac{1}{2}\pi$, $x = 0$, we have $s = 0$, and thus

$$x = r \cos \mu^{\frac{1}{2}} t, \quad \text{.........................(17)}$$

which agrees with the value found in § 2, if $\omega^2 = \mu$. The velocity u is given by

$$u = dx/dt = -r\mu^{\frac{1}{2}} \sin \mu^{\frac{1}{2}} t + s\mu^{\frac{1}{2}} \cos \mu^{\frac{1}{2}} t, \quad \text{.........(18)}$$

and, when $s = 0$,

$$u = -r\mu^{\frac{1}{2}} \sin \mu^{\frac{1}{2}} t, \quad \text{.....................(19)}$$

which agrees with the result in § 3.

Whatever the values of r and s, the velocity, dx/dt, and the displacement, x, both go through complete cycles in the time $2\pi/\sqrt{\mu}$, since, when t increases by $2\pi/\sqrt{\mu}$, the quantity $\mu^{\frac{1}{2}}t$ increases by 2π. Hence the periodic time is $2\pi/\sqrt{\mu}$.

It should be noticed that this solution is applicable to any coördinate and is not limited to the case in which a point moves along a straight line. Thus, if ϕ be any coördinate which fixes the position of the body and if there be a restoring acceleration of the corresponding type of the amount $\mu\phi$, the equation of motion will be

$$\frac{d^2\phi}{dt^2} = -\mu\phi, \quad \text{.......................(20)}$$

and the periodic time will be $2\pi/\sqrt{\mu}$ as before.

11. Angular vibrations. In many cases of oscillation, the body, whose motion is under consideration, instead of moving along a straight line, turns about a fixed axis. Here the position of the body is determined by means of a plane containing the axis and fixed in the body, and, as the body vibrates, this plane vibrates through equal angles on either side of a plane containing the axis and fixed in space. If the angular acceleration, i.e. the rate of change of the angular velocity of the moving plane, be proportional to the angle through which it has turned from the reference plane, and if it always tend to bring the moving plane back to the reference plane, then the motion of the body is again called harmonic. We may speak of the acceleration as a restoring acceleration.

If we take a point N moving on a straight line in such a way that when the moving plane has turned through an angle θ radians from the reference plane in the positive direction, the point N has moved θ cm. in the positive direction from a fixed point O on the line, then the acceleration of N will be numerically equal to the angular acceleration of the moving plane. Hence if the body oscillate under the action of a restoring acceleration $\mu\theta$, the point N will have a restoring acceleration $\mu\theta$ along the straight line. By § 9 the periodic time of N is $2\pi/\sqrt{\mu}$; thus the periodic time of the moving plane is also $2\pi/\sqrt{\mu}$. Since the vibrations of N are isochronous, so are also the vibrations of the body.

Hence, if a body turning about a fixed axis have a restoring acceleration $\mu\theta$, when the body has turned through an angle θ from a zero position, the body will vibrate harmonically and isochronously about that position in the periodic time T, where

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{(\text{angular acceleration for one radian displacement})^{\frac{1}{2}}}.$$

If T be observed, the value of μ can be found from the equation

$$\mu = \frac{4\pi^2}{T^2}. \quad \text{.........................(21)}$$

It will be seen that the above argument applies to any coördinate which fixes the position of any system. If ϕ be such a coördinate and if there be a restoring acceleration corresponding to the coördinate of the amount $\mu\phi$, then the system will have vibrations corresponding to ϕ which are harmonic and isochronous, and have the periodic time $2\pi/\sqrt{\mu}$.

12. Application of dynamics. So far we have been concerned only with kinetics and have merely considered the motion of a system without enquiring *how* that motion has been caused. But, in order to make use of harmonic motion for the determination of some physical quantity Q, we must introduce dynamics and must calculate the acceleration of the system corresponding to some coördinate x in terms of x and Q. If f be the acceleration, we shall find that f is proportional to x (since by supposition the motion is harmonic) and thus we shall be able to write

$$f = -\mu x,$$

where μ is a quantity depending upon Q, but independent of x, at any rate when the amplitude is very small*.

The periodic time T is found by actual observation, and then, by § 9 or § 11, μ can be determined by the equation

$$\mu = \frac{4\pi^2}{T^2},$$

and from this value of μ we can calculate Q.

Instances of this process occur frequently, but the two simple examples given in §§ 13, 14 may aid the reader in applying the process to actual observations.

13. Example (i). A mass M grammes is suspended by a helical spring from a fixed support and the periodic time of the vertical oscillations of M is found to be T seconds. Let us find the restoring force which acts on M, when M is displaced from its equilibrium position through one centimetre, the motion being assumed to be harmonic.

By § 12, the restoring acceleration, when the displacement is x centimetres, is μx cm. sec.$^{-2}$, where

$$\mu = \frac{4\pi^2}{T^2}.$$

Since the mass is M grammes, the restoring force is M times the acceleration and is thus equal to $\mu x M$ dynes. The restoring force is thus proportional to the displacement.

Hence, if F dynes be the restoring force which acts on M when the displacement is one centimetre,

$$F = \mu M = \frac{4\pi^2 M}{T^2} \text{ dynes.}$$

If M be 1066 grammes and T be 1·91 sec., we find that the restoring force for a displacement of one centimetre is

$$F = \frac{4\pi^2 \times 1066}{1{\cdot}91^2} = 11536 \text{ dynes.}$$

14. Example (ii). A body suspended by a vertical wire is found to vibrate about the axis of the wire in the periodic time T secs., the moment of inertia of the body about the axis of the

* It can be shown that, except under circumstances which are mathematically conceivable but would not occur in any experiment, if the vibration is isochronous, so that the periodic time is independent of the amplitude, the motion is harmonic

wire being K gm. cm.². Let us find the restoring couple which the wire exerts on the body, when the body is displaced from its equilibrium position through one radian.

By § 12, the restoring angular acceleration, when the displacement is θ radians, is $\mu\theta$ radians per second per second, where

$$\mu = \frac{4\pi^2}{T^2}.$$

Since the moment of inertia of the body is K gm. cm.², the restoring couple is K times the angular acceleration*, and is thus equal to $\mu\theta K$ dyne-cm. The restoring couple is therefore proportional to the displacement.

Hence, if G dyne-cm. be the restoring couple which the wire exerts on the body when the displacement is one radian,

$$G = \mu K = \frac{4\pi^2 K}{T^2} \text{ dyne-cm.}$$

If the body be a disc 20 cm. in diameter, with its axis vertical, having a mass of 275 grammes, K is $\frac{1}{2} \times 275 \times 10^2$, or 13750 gm. cm.², and thus if T be 1·83 seconds, we find that the restoring couple, when the displacement is one radian, is

$$G = \frac{4\pi^2 \times 13750}{1{\cdot}83^2} = 1{\cdot}6209 \times 10^5 \text{ dyne-cm.}$$

15. Systems with one degree of freedom. In many cases a vibrating system has only one degree of freedom. By this we mean that the configuration of the system is known as soon as a single quantity, which we call a coördinate, is known. As an

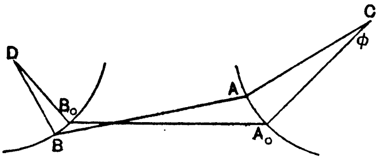

Fig. 3.

example, consider a uniform bar AB (Fig. 3) suspended by two strings from the fixed points C, D. If the system be displaced from its equilibrium position A_0B_0, *the strings remaining in a vertical plane*, the points A, B move on the circles AA_0, BB_0

* *Experimental Elasticity*, Note III.