1.

ANALYTICAL DEVELOPMENT OF FRESNEL'S OPTICAL THEORY OF CRYSTALS.

[*Philosophical Magazine*, XI. (1837), pp. 461—469, 537—541; XII. (1838), pp. 73—83, 341—345.]

THE following is, I believe, the first successful attempt to obtain the full development of Fresnel's Theory of Crystals by direct geometrical methods. Hitherto little has been done beyond finding and investigating the properties of the wave surface, a subject certainly curious and interesting, but not of chief importance for ordinary practical purposes. Mr Kelland, in a most valuable contribution to the *Cambridge Philosophical Transactions**, has incidentally obtained the difference of the squares of the velocities of a plane front in terms of the angles made by it with the optic axes. I have obtained each of the velocities *separately*, and in a form precisely the same for biaxal as for uniaxal crystals.

I have also assigned in my last proposition the place of the lines of vibration in terms of the like quantities, and *that* in a shape remarkably convenient for determining the *plane* of polarization when the ray is given. For at first sight there appears to be some ambiguity in selecting *which* of the *two* lines of vibration is to be chosen when the front is known. If p be the perpendicular from the centre of the surface of elasticity let fall upon the front, ι_1, ι_2 the angles made by the front with the optic planes, ϵ_1, ϵ_2 the angles between its *due* line of vibration and the optic axes, I have shown that

$$\cos \epsilon_1 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2} \cdot \frac{\sin \iota_1}{\sin \iota_2}\right)}, \quad \cos \epsilon_2 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2} \cdot \frac{\sin \iota_2}{\sin \iota_1}\right)},$$

so that all doubt is completely removed. The equation preparatory to obtaining the wave surface is found in Prop. 6 by common algebra, without any use of the properties of maxima and minima, and various other curious relations are discussed.

Without the most careful attention to preserve pure symmetry, the expressions could never have been reduced to their present simple forms.

* See *Lond. and Edinb. Phil. Mag.* Vol. x. p. 336.

ANALYTICAL REDUCTION OF FRESNEL'S OPTICAL THEORY OF CRYSTALS.

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In Proposition 1, a plane front within a crystal being given, the two lines of vibration are investigated.

In Proposition 2 it is shown that the product of the cosines of the inclinations of one of the axes of elasticity to the two lines of vibration, is to the same for either other axis of elasticity in a constant ratio for the same crystal; and the two lines of vibration are proved to be perpendicular to each other.

In Proposition 3, a line of vibration being given, the front to which it belongs is determined; and it is proved that there is only one such, and consequently any line of vibration has but one other line conjugate to it.

In Proposition 4, certain relations are instituted between the positions of, and velocities due to, conjugate lines.

In Proposition 5, the angles made by the front with the planes of elasticity are found in terms of the velocities only.

In Proposition 6, the above is reversed.

In Proposition 7, the position of the planes in which the two velocities are equal (viz. the optic planes) is determined.

In Proposition 8, the position of a front in respect to the optic axes is expressed in terms of the velocities.

In Proposition 9, the problem is reversed, and it is shown that if v_1, v_2 be the two normal velocities with which any front can move perpendicular to itself, and ι_1, ι_2 the angles which it makes with the optic planes, then

$$v_1^2 = a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2,$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2.$$

In the 10th the angle made by a line of vibration with the axes ot elasticity is expressed in terms of the two velocities of the front to which it belongs.

In the 11th Proposition the velocity due to any line of vibration is expressed in terms of the angles which it makes with the optic axes, viz.

$$v^2 - b^2 = (a^2 - c^2)\cos\epsilon_1\cos\epsilon_2.$$

In the 12th Proposition ϵ_1, ϵ_2 are separately expressed in terms of ι_1, ι_2.

In the Appendix I have given the polar or rather radio-angular equation to the wave surface, from which the celebrated proposition of the ray flows as an immediate consequence.

PROPOSITION 1.

If $$lx + my + nz = 0 \qquad (a)$$
be the equation to a given front, to determine the lines of vibration therein.

It is clear that if x, y, z be any point in one of these lines, the force acting on a particle placed there when resolved into the plane must tend to the centre. Consequently the line of force at x, y, z must meet the perpendicular drawn upon the front from the origin. Now the equation to this perpendicular is

$$\frac{X}{l} = \frac{Y}{m} = \frac{Z}{n} \qquad (1)$$

and the forces acting at x, y, z are a^2x, b^2y, c^2z parallel to x, y, z, so that the equation to the line of force is

$$\frac{X - x}{a^2x} = \frac{Y - y}{b^2y} = \frac{Z - z}{c^2z}. \qquad (2)$$

From (2) we obtain

$$b^2yX - a^2xY = (b^2 - a^2)\,xy \qquad (3)$$

$$c^2zY - b^2yZ = (c^2 - b^2)\,yz \qquad (4)$$

$$a^2xZ - c^2zX = (a^2 - c^2)\,zx. \qquad (5)$$

Hence

$$(b^2 - a^2)\,xyn + (c^2 - b^2)\,yzl + (a^2 - c^2)\,zxm = b^2y\,(nX - lZ) + c^2z\,(lY - mX) + a^2x\,(mZ - nY);$$

but by equations (1)

$$lZ - nX = 0, \;\; mX - lY = 0, \;\; nY - mZ = 0$$

therefore

$$(b^2 - a^2)\frac{n}{z} + (c^2 - b^2)\frac{l}{x} + (a^2 - c^2)\frac{m}{y} = 0. \qquad (b)$$

Also we have

$$nz + lx + my = 0 \qquad (a)$$

therefore

$$(b^2 - a^2)\,n^2 + (c^2 - b^2)\,l^2 + nl\left((c^2 - b^2)\frac{z}{x} + (b^2 - a^2)\frac{x}{z}\right) = (a^2 - c^2)\,m^2$$

or

$$(c^2 - b^2)\left(\frac{z}{x}\right)^2 + \frac{1}{nl}\{(c^2 - b^2)\,l^2 + (b^2 - a^2)\,n^2 - (a^2 - c^2)\,m^2\}\frac{z}{x} + (b^2 - a^2) = 0.$$

And in like manner interchanging b, y, m with c, z, n

$$(b^2 - c^2)\left(\frac{y}{x}\right)^2 + \frac{1}{ml}\{(b^2 - c^2)\,l^2 + (c^2 - a^2)\,m^2 - (a^2 - b^2)\,n^2\}\frac{y}{x} + (c^2 - a^2) = 0.$$

Hence if $\left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right)\left(\frac{y_2}{x_2}, \frac{z_2}{x_2}\right)$ be the two systems of values of $\frac{y}{x}, \frac{z}{x}$, then

$$\left(\frac{Y}{X}=\frac{y_1}{x_1},\quad \frac{Z}{X}=\frac{z_1}{x_1}\right)\left(\frac{Y}{X}=\frac{y_2}{x_2},\quad \frac{Z}{X}=\frac{z_2}{x_2}\right)$$

are the two lines of vibration required.

PROPOSITION 2.

By last proposition it appears that

$$\frac{y_1y_2}{x_1x_2}=\frac{c^2-a^2}{b^2-c^2} \qquad (c)$$

and

$$\frac{z_1z_2}{x_1x_2}=\frac{b^2-a^2}{c^2-b^2} \qquad (d)$$

therefore

$$\frac{y_1y_2+z_1z_2}{x_1x_2}=\frac{c^2-b^2}{b^2-c^2}=-1$$

therefore

$$x_1x_2+y_1y_2+z_1z_2=0.$$

And therefore the two lines of vibration are perpendicular to each other.

N.B. Equations (c) and (d) must not be overlooked.

PROPOSITION 3.

A line of vibration is given $\left(\textit{that is } \frac{y_1}{x_1}, \frac{z_1}{x_1} \textit{ are given}\right)$ *and the position of the front is to be determined.*

Let $lx+my+nz=0$ be the front required, then $lx_1+my_1+nz_1=0$, and

$$(b^2-c^2)\frac{l}{x_1}+(c^2-a^2)\frac{m}{y_1}+(a^2-b^2)\frac{n}{z_1}=0.$$

Eliminating n we get

$$l\left((a^2-b^2)\frac{x_1}{z_1}-(b^2-c^2)\frac{z_1}{x_1}\right)+m\left((a^2-b^2)\frac{y_1}{z_1}-(c^2-a^2)\frac{z_1}{y_1}\right)=0$$

therefore

$$\frac{l}{m}=\frac{x_1}{y_1}\frac{(a^2-b^2)\,y_1^2-(c^2-a^2)\,z_1^2}{(b^2-c^2)\,z_1^2-(a^2-b^2)\,x_1^2}$$

$$=\frac{x_1}{y_1}\frac{a^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}{b^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}.$$

If now we make $x_1^2 + y_1^2 + z_1^2 = 1$

$$a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = v_1^2$$

and therefore

$$\frac{l}{m} = \frac{x_1}{y_1} \cdot \frac{a^2 - v_1^2}{b^2 - v_1^2}$$

and in like manner

$$\frac{l}{n} = \frac{x_1}{z_1} \cdot \frac{a^2 - v_1^2}{c^2 - v_1^2};$$

therefore

$$(a^2 - v_1^2)\, x_1x + (b^2 - v_1^2)\, y_1y + (c^2 - v_1^2)\, z_1z = 0$$

is the equation required.

PROPOSITION 4.

$\frac{l}{m}, \frac{l}{n}$ having each only one value, shows that only one front corresponds to the given line of vibration. Let x_2, y_2, z_2, v_2 correspond to x_1, y_1, z_1, v_1 for the conjugate line of vibration, then the equation to the front may be expressed likewise by

$$(a^2 - v_2^2)\, x_2x + (b^2 - v_2^2)\, y_2y + (c^2 - v_2^2)\, z_2z = 0,$$

so that

$$\frac{(a^2 - v_1^2)\, x_1}{(a^2 - v_2^2)\, x_2} = \frac{(b^2 - v_1^2)\, y_1}{(b^2 - v_2^2)\, y_2} = \frac{(c^2 - v_1^2)\, z_1}{(c^2 - v_2^2)\, z_2}.$$

PROPOSITION 5.

To find ω, ϕ, ψ, the angles made by the front with the planes of elasticity in terms of v_1, v_2.

By the last proposition

$$(\cos\omega)^2 = \frac{(a^2 - v_1^2)^2\, x_1^2}{(a^2 - v_1^2)^2\, x_1^2 + (b^2 - v_1^2)^2\, y_1^2 + (c^2 - v_1^2)^2\, z_1^2}$$

$$= \frac{(a^2 - v_1^2)\,(a^2 - v_2^2)\, x_1x_2}{(a^2 - v_1^2)\,(a^2 - v_2^2)\, x_1x_2 + (b^2 - v_1^2)\,(b^2 - v_2^2)\, y_1y_2 + (c^2 - v_1^2)\,(c^2 - v_2^2)\, z_1z_2}.$$

Now, by Proposition 2,

$$\frac{x_1x_2}{c^2 - b^2} = \frac{y_1y_2}{a^2 - c^2} = \frac{z_1z_2}{b^2 - a^2}$$

therefore $(\cos\omega)^2$

$$= \frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)+(b^2-v_1^2)(b^2-v_2^2)(a^2-c^2)+(c^2-v_1^2)(c^2-v_2^2)(b^2-a^2)}$$

$$= \frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{a^4(c^2-b^2)+b^4(a^2-c^2)+c^4(a^2-b^2)}$$

$$= \frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly,

$$(\cos\phi)^2 = \frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)},$$

$$(\cos\psi)^2 = \frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}.$$

PROPOSITION 6.

To find v_1, v_2 in terms of ω, ϕ, ψ.

By the last proposition

$$\frac{(\cos\omega)^2}{a^2-v_1^2} = \frac{a^2}{(a^2-b^2)(a^2-c^2)} - v_2^2 \cdot \frac{1}{(a^2-b^2)(a^2-c^2)}$$

$$\frac{(\cos\phi)^2}{b^2-v_1^2} = \frac{b^2}{(b^2-a^2)(b^2-c^2)} - v_2^2 \cdot \frac{1}{(b^2-a^2)(b^2-c^2)}$$

$$\frac{(\cos\psi)^2}{c^2-v_1^2} = \frac{c^2}{(c^2-b^2)(c^2-a^2)} - v_2^2 \cdot \frac{1}{(c^2-a^2)(c^2-b^2)}$$

therefore

$$\frac{(\cos\omega)^2}{a^2-v_1^2} + \frac{(\cos\phi)^2}{b^2-v_1^2} + \frac{(\cos\psi)^2}{c^2-v_1^2} = 0.$$

Just in the same way

$$\frac{(\cos\omega)^2}{a^2-v_2^2} + \frac{(\cos\phi)^2}{b^2-v_2^2} + \frac{(\cos\psi)^2}{c^2-v_2^2} = 0,$$

so that v_1^2, v_2^2 are the two roots of the equation

$$\frac{(\cos\omega)^2}{a^2-v^2} + \frac{(\cos\phi)^2}{b^2-v^2} + \frac{(\cos\psi)^2}{c^2-v^2} = 0.$$

COR. Hence the equation to the wave surface may be obtained by making

$$(\cos\omega)\,x + (\cos\phi)\,y + (\cos\psi)\,z = v,$$

or if we please to apply Prop. 5, we may make

$$\sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}}\,.\,x+\sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}}\,.\,y$$
$$+\sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}\,.\,z=v_1,$$

or, if we please*,

$$\sqrt{\frac{(a^2u^2-1)(a^2-v^2)}{(a^2-b^2)(a^2-c^2)}}\,.\,x+\sqrt{\frac{(b^2u^2-1)(b^2-v^2)}{(b^2-a^2)(b^2-c^2)}}\,.\,y$$
$$+\sqrt{\frac{(c^2u^2-1)(c^2-v^2)}{(c^2-a^2)(c^2-b^2)}}\,.\,z=1.$$

Proposition 7.

To find when $v_1=v_2$.

By Prop. 4,

$$\frac{x_1(v_1^2-a^2)}{x_2(v_2^2-a^2)}=\frac{y_1(v_1^2-b^2)}{y_2(v_2^2-b^2)}=\frac{z_1(v_1^2-c^2)}{z_2(v_2^2-c^2)}. \qquad (\theta)$$

Hence when $v_1=v_2$ we have, generally speaking,

$$\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}.$$

Now $$x_1x_2+y_1y_2+z_1z_2=0;$$

therefore $x_1^2+y_1^2+z_1^2$ would $=0$, which is absurd.

The only case therefore when v_1 can $=v_2$ is when one of those terms of equation (θ) becomes $\frac{0}{0}$: thus suppose $v_1=b$, then we have $\frac{x_1}{x_2}=\frac{z_1}{z_2}=\frac{0}{0}$, and we can no longer infer $\frac{x_1}{x_2}=\frac{y_1}{y_2}$.

Let now $(\omega_1, \phi_1, \psi_1)(\omega_2, \phi_2, \psi_2)$ be the two systems of values which ω, ϕ, ψ assume when $v_1=v_2=b$, then applying the equation of Prop. 5 we have

$$\cos\omega_1=\sqrt{\frac{a^2-b^2}{a^2-c^2}} \qquad \cos\omega_2=\sqrt{\frac{a^2-b^2}{a^2-c^2}}$$
$$\cos\phi_1=0 \qquad \cos\phi_2=0$$
$$\cos\psi_1=\sqrt{\frac{b^2-c^2}{a^2-c^2}} \qquad \cos\psi_2=\sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

so that b must correspond to the mean axis.

[* See below, p. 27. Ed.]

PROPOSITION 8.

ι_1, ι_2 being the angles made by the front with the optic planes, to find ι_1, ι_2 in terms of v_1, v_2.

By analytical geometry

$$\cos\iota_1 = \cos\omega \,.\, \cos\omega_1 + \cos\phi \,.\, \cos\phi_1 + \cos\psi \,.\, \cos\psi_1$$

$$= \sqrt{\frac{(v_1^2-a^2)(v_2^2-a^2)}{(a^2-b^2)(a^2-c^2)}} \,.\, \sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

$$+ \sqrt{\frac{(v_1^2-c^2)(v_2^2-c^2)}{(c^2-a^2)(c^2-b^2)}} \,.\, \sqrt{\frac{c^2-b^2}{c^2-a^2}}$$

$$= \frac{\sqrt{\{(v_1^2-a^2)(v_2^2-a^2)\}} + \sqrt{\{(v_1^2-c^2)(v_2^2-c^2)\}}}{a^2-c^2},$$

and similarly

$$\cos\iota_2 = \cos\omega \,.\, \cos\omega_2 + \cos\phi \,.\, \cos\phi_2 + \cos\psi \,.\, \cos\psi_2$$

$$= \frac{\sqrt{\{(v_1^2-a^2)(v_2^2-a^2)\}} - \sqrt{\{(v_1^2-c^2)(v_2^2-c^2)\}}}{a^2-c^2}.$$

PROPOSITION 9.

To find v_1, v_2 in terms of ι_1, ι_2.

By the last proposition

$$\cos\iota_1 \,.\, \cos\iota_2 = \frac{(v_1^2-a^2)(v_2^2-a^2) - (v_1^2-c^2)(v_2^2-c^2)}{(a^2-c^2)^2}$$

$$= \frac{(a^4-c^4) - (a^2-c^2)(v_1^2+v_2^2)}{(a^2-c^2)^2}$$

$$= \frac{(a^2+c^2) - (v_1^2+v_2^2)}{(a^2-c^2)}$$

therefore

$$v_1^2 + v_2^2 = a^2 + c^2 - (a^2-c^2)\cos\iota_1\cos\iota_2.$$

Again, $(\sin\iota_1)^2 \,.\, (\sin\iota_2)^2 = 1 - (\cos\iota_1)^2 - (\cos\iota_2)^2 + (\cos\iota_1)^2(\cos\iota_2)^2$

$$= 1 - 2\,.\,\frac{(v_1^2-a^2)(v_2^2-a^2) + (v_1^2-c^2)(v_2^2-c^2)}{(a^2-c^2)^2}$$

$$+ \frac{(a^2+c^2)^2 - 2(a^2+c^2)(v_1^2+v_2^2) + (v_1^2+v_2^2)^2}{(a^2-c^2)^2}$$

$$= \frac{v_1^4 - 2v_1^2v_2^2 + v_2^4}{(a^2-c^2)^2}$$

therefore

$$v_1^2 - v_2^2 = (a^2 - c^2) \sin \iota_1 . \sin \iota_2$$

but $$v_1^2 + v_2^2 = (a^2 + c^2) - (a^2 - c^2) \cos \iota_1 \cos \iota_2$$

therefore

$$v_1^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 + \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 + \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 + \iota_2}{2}\right)^2$$

$$v_2^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 - \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 - \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 - \iota_2}{2}\right)^2 .$$

Thus for uniaxal crystals where $\iota_1 + \iota_2 = 180°$

$$v_1^2 = a^2$$

$$v_2^2 = a^2 (\cos \iota)^2 + c^2 (\sin \iota)^2.$$

Cor. Hence we may reduce the discovery of the two fronts into which a plane front is refracted on entering a crystal to the following trigonometrical problem.

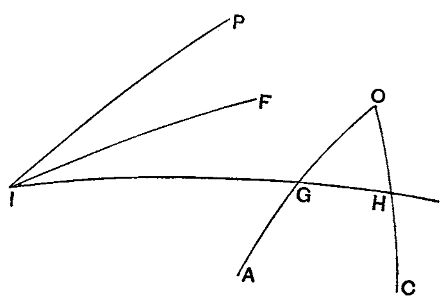

Fig. 1.

Let a sphere be described about any point in the line in which the air front intersects the plane of incidence. Let the great circle PI denote the latter plane, IF the former, OA, OC also great circles, the planes of single velocity. Suppose IGH to be one of the refracted fronts intersecting OA, OC in G and H, then

$$\frac{(a^2 + c^2) - (a^2 - c^2) \cos (G + H)}{2\,(\text{vel. in air})^2} = \frac{(\sin PIF)^2}{(\sin PIGH)^2} .$$

The double sign will give rise to two positions of the refracted front IGH.

The propositions which follow are perhaps more curious than immediately useful.

PROPOSITION 10.

To determine the portion of a line of vibration in terms of the two velocities of its corresponding front.

We have here to determine the quantities $\frac{y_1}{x_1}, \frac{z_1}{x_1}$ (of Prop. 1) in terms of v_1, v_2, or on putting $x_1^2+y_1^2+z_1^2=1$, x_1, y_1, z_1 are to be found in terms of v_1, v_2.

By Prop. 3

$$x_1 : y_1 : z_1 :: \frac{l}{a^2-v_1^2} : \frac{m}{b^2-v_1^2} : \frac{n}{c^2-v_1^2}$$

and by Prop. 5

$$\begin{aligned} l^2 : m^2 : n^2 :: &(b^2-c^2)(a^2-v_1^2)(a^2-v_2^2) \\ &: (c^2-a^2)(b^2-v_1^2)(b^2-v_2^2) \\ &: (a^2-b^2)(c^2-v_1^2)(c^2-v_2^2); \end{aligned}$$

therefore $\quad x_1^2 \quad : \quad y_1^2 \quad : \quad z_1^2$

$$:: (b^2-c^2)\frac{a^2-v_2^2}{a^2-v_1^2} : (c^2-a^2)\frac{b^2-v_2^2}{b^2-v_1^2} : (a^2-b^2)\frac{c^2-v_2^2}{c^2-v_1^2}.$$

Let α, β, γ be the angles made by the given line of vibration with the elastic axes, then

$$(\cos\alpha)^2 = \frac{x_1^2}{x_1^2+y_1^2+z_1^2}$$

$$= (b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)$$

divided by

$$(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2) + (c^2-a^2)(b^2-v_2^2)(c^2-v_1^2)(a^2-v_1^2) \\ + (a^2-b^2)(c^2-v_2^2)(a^2-v_1^2)(b^2-v_1^2)$$

and therefore

$$= \frac{(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)}{(v_1^2-v_2^2)(a^2-b^2)(b^2-c^2)(c^2-a^2)}$$

(where it is to be observed that the reduction of the denominator is simply the effect of a vast heap of terms disappearing under the influence of contact with the magic circuit (a^2-b^2), (b^2-c^2), (c^2-a^2), a simpler instance of which was seen in Proposition 5).

In fact the coefficient of $v^4 . v^2$

$$= (b^2-c^2)+(c^2-a^2)+(a^2-b^2)$$

$$= 0$$