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978-1-107-65024-4 - Theory of Differential Equations: Part I: Exact Equations and Pfaff's Problem

Andrew Russell Forsyth

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