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THEORY
OF
DIFFERENTIAL EQUATIONS.
PART I.
EXACT EQUATIONS AND PFAFF'S PROBLEM.

BY

ANDREW RUSSELL FORSYTH, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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PREFACE.

THE present volume is the first contribution towards the fulfilment of a promise made at the time of publication of my *Treatise on Differential Equations*. My desire has been to include every substantial contribution to the development of the particular subject herein dealt with; and the historical form, into which the treatment has been cast, has facilitated the indication of the continuous course of the development.

All sources of information, which have been drawn upon, are quoted in their proper connection; a few investigations have been added, which I believe to be new; and some examples have been made, in order to provide illustrations of various methods.

In the revision of the proof-sheets I have had, and wish to acknowledge most gratefully, the valuable assistance of my friend Mr. H. M. Taylor, Fellow of Trinity College, Cambridge. The volume owes much to the care and trouble he has ungrudgingly bestowed upon it. My thanks are also due to Mr. H. F. Baker, Fellow of St. John's College, Cambridge, for his kindness in reading the proof-sheets.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE,
28 July, 1890.

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