CHAPTER I.

SINGLE EXACT EQUATION*.

1. WHEN a number of variables $x, y, z, u, \ldots$ are connected by a permanent relation of the form

$$\phi(x, y, z, u, \ldots) = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a is a constant, any simultaneous small variations $dx, dy, dz, du, \ldots$ to which the variables are subjected are so related that the equation

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \frac{\partial\phi}{\partial u}du + \ldots = 0$$

is satisfied; and, if a relation between the small variations be given in this form, the equivalent integral relation is at once obtained in the form of the first equation.

The equation which connects these small variations is exact, for its left-hand member is a complete differential. But if the first differential coefficients of ϕ with regard to the variables have a common factor μ so that we may take

$$\frac{\partial\phi}{\partial x} = \mu P, \quad \frac{\partial\phi}{\partial y} = \mu Q, \quad \frac{\partial\phi}{\partial z} = \mu R, \quad \frac{\partial\phi}{\partial u} = \mu S, \ldots\ldots$$

* It is to be understood that the investigations in the first chapter relative to exact linear equations are additional to the very slight sketch of such equations given in §§ 150—164 of my *Treatise on Differential Equations*, hereafter referred to as *Treatise*; and that the investigations in the second chapter relative to the integration of systems of partial differential equations are intended specially to indicate Mayer's theory of a system of equations of a particular form and to be supplementary to the investigations of Bour and Jacobi.

then the relation connecting the variations becomes

$$Pdx + Qdy + Rdz + Sdu + \ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots(2),$$

on the removal of the factor μ which is not dependent on these variations. This new equation (2) is essentially the same as the earlier equation; but it is not necessarily, and in general it is not, an exact equation. In order to be made an exact equation so that the integral relation (1) may be deduced, the factor μ, which may be called the integrating factor, must be restored; and, as no indication of the form of μ survives in the reduced equation, the determination of the factor must be made by a separate investigation.

2. There are equivalent forms of (1) which lead to the same equation (2). Let Φ be any function of ϕ, say

$$\Phi = f(\phi);$$

then, if c be the value of $f(a)$, the equation (1) may be replaced by

$$\Phi = c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)',$$

where Φ is a function of $x, y, z, u, \ldots$ and c is a constant. The same small variations to which the variables are subjected are now connected by the equation

$$\frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz + \frac{\partial \Phi}{\partial u} du + \ldots = 0.$$

But, since $x, y, z, u, \ldots$ enter into Φ only through ϕ, we have

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial \phi} \mu P = P\mu f'(\phi),$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial \phi} \frac{\partial \phi}{\partial y} = \frac{\partial \Phi}{\partial \phi} \mu Q = Q\mu f'(\phi),$$

with similar equations; and therefore the equation connecting the variations reduces to (2) as before on the removal of the factor M, where

$$M = \mu f'(\phi);$$

and therefore M is an integrating factor which will enable us to obtain the equation (1)$'$. Hence for every form of f leading to an integral equation new in form, we have a corresponding integrating factor.

It is convenient to call a function of the variables a *solution* of (2), if (2) be satisfied in virtue of the relation obtained by equating that function to a constant; thus ϕ and Φ are solutions of (2). The result just obtained shews that, *if two quantities be functions of one another, they are solutions of the same equation.*

3. Conversely, if the differential equation

$$Pdx + Qdy + Rdz + Sdu + \ldots = 0$$

(assumed to be the only relation connecting the differentials of the variables) can be satisfied in virtue of a single integral equation *all its solutions are equivalent to one another*, that is, one solution is sufficient for the construction of all the solutions. For let

$$\phi = \phi(x, y, z, u, \ldots\ldots)$$

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be two solutions, so that we have

$$d\phi = 0, \quad d\Phi = 0.$$

When x is eliminated between the two integral equations, the resulting equation is of the form

$$F(\phi, \Phi, y, z, u, \ldots\ldots) = 0.$$

Hence

$$\frac{\partial F}{\partial \phi} d\phi + \frac{\partial F}{\partial \Phi} d\Phi + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial u} du + \ldots = 0,$$

so that

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial u} du + \ldots = 0,$$

a differential equation among the same variables and distinct from the original differential equation because the variation dx does not occur. But as the original equation is the only relation connecting the differentials of the variables, it follows that the new equation, not satisfied in virtue of that original equation, is evanescent; and therefore

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial u} = 0, \ldots\ldots$$

so that F is explicitly independent of $y, z, u, \ldots$ and takes the form

$$F(\Phi, \phi) = 0.$$

Hence Φ can be obtained in terms of ϕ, which proves the proposition.

4. Again the integrating factors M and μ corresponding to Φ and ϕ are such that

$$M \div \mu = f'(\phi),$$

and $f'(\phi)$, a function of the solution ϕ, is (§ 2) a solution; hence the *quotient of two integrating factors, if not a constant, is a solution of the equation.*

And, *if ϕ be the solution determined by the factor μ, every other factor is of the form $\mu \, . \, \lambda(\phi)$ where $\lambda(\phi)$ is a function of ϕ.*

5. When a differential equation of the form at present under consideration is given, there is not necessarily a single integral equation in virtue of which it is satisfied. The conditions that this may be so are that the coefficients $P, Q, R, S, \ldots$ of the differentials are proportional to the partial differential coefficients of one function, conditions which are not satisfied for any set of arbitrarily assigned quantities $P, Q, R, S, \ldots$; and these conditions lead to relations between the quantities, which must be satisfied in order that the differential equation may have a single equation as its integral equivalent. We proceed to obtain these relations.

Let the differential equation be

$$X_1 dx_1 + X_2 dx_2 + \ldots\ldots\ldots + X_p dx_p = 0 \ldots\ldots\ldots\ldots(3),$$

and suppose it derived from the equation

$$\phi(x_1, x_2, \ldots\ldots, x_p) = \text{constant} \ldots\ldots\ldots\ldots\ldots(4),$$

by the rejection of a factor μ after differentiation; then we have

$$\mu X_r = \frac{\partial \phi}{\partial x_r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

for values 1, 2, ..., p of r. From the equations (5) it follows that

$$\frac{\partial}{\partial x_m}(\mu X_n) = \frac{\partial^2 \phi}{\partial x_m \partial x_n} = \frac{\partial}{\partial x_n}(\mu X_m)$$

for any two indices m and n; and therefore

$$X_n \frac{\partial \mu}{\partial x_m} - X_m \frac{\partial \mu}{\partial x_n} = \mu \left(\frac{\partial X_m}{\partial x_n} - \frac{\partial X_n}{\partial x_m} \right)$$

$$= \mu a_{m,n},$$

where

$$a_{m,n} = \frac{\partial X_m}{\partial x_n} - \frac{\partial X_n}{\partial x_m} \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

If r denote any other index, we have similarly

$$X_r \frac{\partial \mu}{\partial x_n} - X_n \frac{\partial \mu}{\partial x_r} = \mu a_{n,r},$$

and

$$X_m \frac{\partial \mu}{\partial x_r} - X_r \frac{\partial \mu}{\partial x_m} = \mu a_{r,m}.$$

Multiplying these three equations by X_r, X_m, X_n respectively and adding, we have

$$0 = \mu\,(a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n),$$

or, since μ does not vanish,

$$a_{m,n} X_r + a_{n,r} X_m + a_{r,m} X_n = 0 \quad\text{..............(7).}$$

This equation, which is evanescent if two of the indices be the same, holds for any combination of three of the indices of the series $1, 2, \ldots, p$; and therefore the number of equations, of the same form as (7), between the quantities X is

$$\tfrac{1}{6}\, p\,(p-1)(p-2),$$

each being identically satisfied.

6. These equations are not, however, all independent of one another. Taking any other index s, distinct from m, n, r we have, in addition to (7),

$$a_{s,m} X_r + a_{r,s} X_m + a_{m,r} X_s = 0 \quad\text{..............(7)}',$$

$$a_{m,s} X_n + a_{s,n} X_m + a_{n,m} X_s = 0 \quad\text{..............(7)}'',$$

and lastly

$$a_{n,r} X_s + a_{r,s} X_n + a_{s,n} X_r = 0 \quad\text{..............(7)}'''.$$

Multiplying (7), (7)′, (7)″ by X_s, X_n, X_r respectively and adding we have, in virtue of the property

$$a_{k,l} = -a_{l,k}$$

for all pairs of indices, the relation

$$X_m\,(a_{n,r} X_s + a_{r,s} X_n + a_{s,n} X_r) = 0,$$

which is, in effect, the equation (7)‴ since X_m does not vanish. Hence of the four equations, each involving three of a set of four indices, only three are independent; any one of the four equations can be deduced from the other three.

Let us consider as the three independent equations those which involve m, n, r; m, r, s; m, s, n; in the foregoing set they are (7), (7)′, (7)″. If between (7)′ and (7)″ we eliminate X_s we have

$$a_{m,n}\, a_{s,m}\, X_r + (a_{m,n}\, a_{r,s} + a_{m,r}\, a_{s,n})\, X_m + a_{r,m}\, a_{s,m}\, X_n = 0\,;$$

to which if (7) multiplied by $a_{m,s}$ be added, we have

$$a_{m,n}\, a_{r,s} + a_{m,r}\, a_{s,n} + a_{m,s}\, a_{n,r} = 0$$

on the rejection of the factor X_m. This last equation is satisfied because (7), (7)′, (7)″ are satisfied; if in any case desirable, it could replace any one of the three.

Since the equation which involves the indices n, r, s is deducible from the three which involve pairs of these indices and some other index the same for the three, we shall obtain all the independent equations by taking some definite index, say 1, and forming all the sets of three. The aggregate of all these sets is really the aggregate obtained by combining the index 1 with every pair of indices other than 1, that is, with every pair formed from $2, 3, \ldots, p$; and the equations in this aggregate are independent of one another. Hence the number of independent equations of condition is

$$\tfrac{1}{2}(p-1)(p-2).$$

It is to be noticed that, if μ be unity, then the equations are all satisfied in virtue of the vanishing of the quantities $a_{m,n}$; and the equations of condition are in this case

$$a_{m,n} = 0,$$

their number being $\tfrac{1}{2}p(p-1)$. The extra number of conditions arises from the additional supposed limitation that the equation is exact as given and therefore requires no factor to make it so.

7. The conditions of the type (7) are a necessary consequence of the supposition that the differential equation (3) can be made an exact differential; it will now be shewn conversely that, *if the conditions* (7) *be satisfied, then the differential equation can be made exact.*

It is known from the theory of equations which involve only two variables x and y that for an equation

$$Pdx + Qdy = 0$$

there exists a function $\theta(x, y)$ such that the differential equation is satisfied in virtue of the relation

$$\theta(x, y) = \text{constant},$$

and therefore that P and Q are proportional to the derivatives of θ with regard to x and y respectively. Considering then X_1 and X_2 as functions of x_1 and x_2, we infer that there exists a function u of x_1 and x_2 such that for some quantity λ we may write

$$\lambda X_1 = \frac{\partial u}{\partial x_1}, \quad \lambda X_2 = \frac{\partial u}{\partial x_2};$$

and the function u will involve the other quantities which occur in X_1 and X_2, viz. $x_3, x_4, \ldots, x_p$, the presence of which does not however affect derivation with regard to x_1 and x_2. But it may not be inferred that the remaining coefficients in the equation are similarly proportional to the remaining derivatives; and we therefore take

$$\lambda X_r - \frac{\partial u}{\partial x_r} = Y_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

(for $r = 3, 4, \ldots, p$), where Y_r may be considered known when u is, as it is supposed to be, known.

These new quantities Y_r will satisfy certain equations, which are derivable in virtue of the aggregate (7). It has been seen that only three of the four equations which involve four indices need be retained in that aggregate; and, as already (§ 6) explained, the retained equations will be taken to be made up of

(i) the $p-2$ equations involving the indices $1, 2, r$,

(ii) the $\frac{1}{2}(p-2)(p-3)$ .... $1, r, s$,

where r and s are different from one another and are terms in the series $3, 4, \ldots, p$. This set of combinations is evidently the set obtained by combining the index 1 with every pair formed from $2, 3, \ldots, p$.

8. Considering the first of the two series of retained equations we have, for each index r,

$$\frac{\partial}{\partial x_1}(\lambda X_r - Y_r) = \frac{\partial^2 u}{\partial x_r \partial x_1} = \frac{\partial}{\partial x_r}(\lambda X_1),$$

so that
$$X_r \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_r} = \lambda a_{1,r} + \frac{\partial Y_r}{\partial x_1}.$$

Similarly
$$X_r \frac{\partial \lambda}{\partial x_2} - X_2 \frac{\partial \lambda}{\partial x_r} = \lambda a_{2,r} + \frac{\partial Y_r}{\partial x_2},$$

and
$$X_2 \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_2} = \lambda a_{1,2}.$$

Now of the aggregate (7) the retained equation which involves the indices 1, 2, r is

$$a_{1,2} X_r + a_{2,r} X_1 + a_{r,1} X_2 = 0;$$

so that, multiplying the preceding equations by $-X_2$, X_1, X_r respectively, adding and using the condition-relation, we have

$$X_1 \frac{\partial Y_r}{\partial x_2} - X_2 \frac{\partial Y_r}{\partial x_1} = 0.$$

This is the only equation of series (i) which involves Y_r alone; all the equations of series (ii) involve two of the quantities Y, and the import of such equations will be indicated immediately. We may thus regard the preceding equation as an equation determining the form of Y_r. It is a linear partial differential equation of the first order; to obtain the most general solution we construct $p-1$ independent integrals of the subsidiary equations

$$\frac{dx_1}{-X_2} = \frac{dx_2}{X_1} = \frac{dx_3}{0} = \frac{dx_4}{0} = \dots\dots = \frac{dx_p}{0}.$$

There are $p-2$ integrals at once given in the form

$$x_r = \text{constant} \quad (r = 3, 4, \dots, p);$$

so that only one more is needed, to be given by

$$X_1 dx_1 + X_2 dx_2 = 0,$$

or $$\lambda \left(\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 \right) = 0,$$

i.e., $$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0.$$

But in the simultaneous system dx_3, dx_4, ... all vanish, and therefore the last equation may be taken in the form

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \dots\dots = 0$$

viz., $$du = 0;$$

and therefore the other required integral is

$$u = \text{constant}.$$

Hence, by the theory of linear partial differential equations, it follows that Y_r is of the form

$$Y_r = f_r(u, x_3, x_4, \ldots, x_p) \qquad \text{(9)},$$

where f_r may at present imply *any* function of the arguments.

Multiplying the equation (3) by λ and substituting from (8) for the quantities λX, the new form of the equation is

$$du + Y_3 dx_3 + Y_4 dx_4 + \ldots\ldots + Y_p dx_p = 0 \qquad \text{(3)}',$$

where the quantities Y_r are given by the equations (9). Hence, in virtue of the first series of retained equations of § 7, *the given differential equation* (3) *has been transformed into another* (3)′ *involving one variable fewer.*

9. Consider now the second of the two series of retained equations of § 7. Taking a typical equation of the series, we have

$$a_{1,r} X_s + a_{r,s} X_1 + a_{s,1} X_r = 0.$$

But
$$\frac{\partial}{\partial x_r}(\lambda X_s - Y_s) = \frac{\partial^2 u}{\partial x_r \partial x_s} = \frac{\partial}{\partial x_s}(\lambda X_r - Y_r),$$

so that
$$X_s \frac{\partial \lambda}{\partial x_r} - X_r \frac{\partial \lambda}{\partial x_s} = \lambda a_{r,s} + \frac{\partial Y_s}{\partial x_r} - \frac{\partial Y_r}{\partial x_s}.$$

And, before, we had

$$X_r \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_r} = \lambda a_{1,r} + \frac{\partial Y_r}{\partial x_1};$$

similarly
$$X_s \frac{\partial \lambda}{\partial x_1} - X_1 \frac{\partial \lambda}{\partial x_s} = \lambda a_{1,s} + \frac{\partial Y_s}{\partial x_1}.$$

Multiplying these three equations by X_1, X_s, $-X_r$ respectively, adding and using the former relation, we have

$$X_1\left(\frac{\partial Y_s}{\partial x_r} - \frac{\partial Y_r}{\partial x_s}\right) + X_s \frac{\partial Y_r}{\partial x_1} - X_r \frac{\partial Y_s}{\partial x_1} = 0.$$

In this equation the quantities Y_r and Y_s are functions of $x_1, x_2, \ldots, x_p$ as given by (8). When we take their forms as determined in (9), we have

$$\frac{\partial Y_r}{\partial x_1} = \frac{\partial f_r}{\partial u}\frac{\partial u}{\partial x_1} = \lambda X_1 \frac{\partial f_r}{\partial u},$$

$$\frac{\partial Y_r}{\partial x_s} = \frac{\partial f_r}{\partial u}\frac{\partial u}{\partial x_s} + \frac{\partial f_r}{\partial x_s} = (\lambda X_s - f_s)\frac{\partial f_r}{\partial u} + \frac{\partial f_r}{\partial x_s},$$

and therefore

$$X_1 \frac{\partial Y_r}{\partial x_s} - X_s \frac{\partial Y_r}{\partial x_1} = X_1 \left(\frac{\partial f_r}{\partial x_s} - f_s \frac{\partial f_r}{\partial u} \right).$$

Similarly $$X_1 \frac{\partial Y_s}{\partial x_r} - X_r \frac{\partial Y_s}{\partial x_1} = X_1 \left(\frac{\partial f_s}{\partial x_r} - f_r \frac{\partial f_s}{\partial u} \right);$$

and therefore the above equation becomes, on the rejection of the non-vanishing factor $-X_1$,

$$\frac{\partial f_r}{\partial x_s} - \frac{\partial f_s}{\partial x_r} + f_r \frac{\partial f_s}{\partial u} - f_s \frac{\partial f_r}{\partial u} = 0 \text{ ....(10)},$$

for all the combinations of the indices r and s. These are the equations derived from the second series of retained equations of § 7.

It thus follows that the coefficients of the transformed equation

$$du + f_3 dx_3 + f_4 dx_4 + \ldots\ldots + f_p dx_p = 0 \text{ ....(3)}',$$

equivalent to (3), are subject to the conditions (10).

10. Since the new equation (3)′ is equivalent to (3), let us find the set of conditions which bear the same relation to (3)′ as the set (7) bear to (3). Associating a subscript index 0 with u and defining $b_{r,s}$ by the equation

$$b_{r,s} = \frac{\partial f_r}{\partial x_s} - \frac{\partial f_s}{\partial x_r}$$

for all pairs of the indices 0, 3, 4, ..., p, the complete set of conditions, associated with (3)′ and similar to (7), are

$$b_{r,s} f_q + b_{s,q} f_r + b_{q,r} f_s = 0,$$

in number equal to $\frac{1}{6}(p-1)(p-2)(p-3)$. But of this number only $\frac{1}{2}(p-2)(p-3)$ are independent; and an independent set, as in the earlier case, can be obtained by retaining all those equations which involve any one index, say 0, with all possible pairs of indices from 3, 4, ..., p. Taking then $q = 0$ we have

$$f_q = 1,$$

$$b_{s,q} = \frac{\partial f_s}{\partial u},$$

$$b_{q,r} = -\frac{\partial f_r}{\partial u};$$