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Excerpt
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0 Introduction: Geometry and Geometries

Geometry is the study of shape. It takes its name from the Greek belief that geometry began with Egyptian surveyors of two or three millennia ago measuring the Earth, or at least the fertile expanse of it that was annually flooded by the Nile.

The word comes from the Greek words *geo* (Earth) and *metria* (measuring).

It rapidly became more ambitious. Classical Greek geometry, called *Euclidean geometry* after Euclid, who organized an extensive collection of theorems into his definitive text *The Elements*, was regarded by all in the early modern world as the true geometry of space. Isaac Newton used it to formulate his *Principia Mathematica* (1687), the book that first set out the theory of gravity. Until the mid-19th Century, Euclidean geometry was regarded as one of the highest points of rational thought, as a foundation for practical mathematics as well as advanced science, and as a logical system splendidly adapted for the training of the mind. We shall see in this book that by the 1850s geometry had evolved considerably – indeed, whole new geometries had been discovered.

Isaac Newton (1643–1727) was an English astronomer, physicist and mathematician. He was Professor of Mathematics at Cambridge, Master of the Royal Mint, and successor of Samuel Pepys as President of the Royal Society.

The idea of using coordinates in geometry can be traced back to Apollonius's treatment of conic sections, written a generation after Euclid. But their use in a systematic way with a view to simplifying the treatment of geometry is really due to Fermat and Descartes. Fermat showed how to obtain an equation in two variables to describe a conic or a straight line in 1636, but his work was only published posthumously in 1679. Meanwhile in 1637 Descartes published his book *Discourse on Method*, with an extensive appendix entitled *La Géometrie*, in which he showed how to introduce coordinates to solve a wide variety of geometrical problems; this idea has become so central a part of mathematics that whole sections of *La Géometrie* read like a modern textbook.

Apollonius of Perga (c. 255–170 BC) was a Greek geometer, whose only surviving work is a text on conics.

Pierre de Fermat (1601–1665) was a French lawyer and amateur mathematician, who claimed to have a proof of the recently proved Fermat's Last Theorem in Number Theory.

A contemporary of Descartes, Girard Desargues, was interested in the ideas of perspective that had been developed over many centuries by artists (anxious to portray three-dimensional scenes in a realistic way on two-dimensional walls or canvases). For instance, how do you draw a picture of a building, or a staircase, which your client can understand and commission, and from which artisans can deduce the correct dimensions of each stone? Desargues also realized that since any two conics can always be obtained as sections of the same cone in $\mathbb{R}^3$, it is possible to present the theory of conics in a unified

René Descartes (1596–1650) was a French scientist, philosopher and mathematician. He is also known for the phrase 'Cogito, ergo sum' (I think, therefore I am).

way, using concepts which later mathematicians distilled into the notion of the cross-ratio of four points. Desargues' discoveries came to be known as *projective geometry*.

Girard Desargues (1591–1661) was a French architect.
We deal with these ideas in Chapters 4 and 5.

Blaise Pascal was the son of a mathematician, Étienne, who attended a group of scholars frequented by Desargues. He heard of Desargues's work from his father, and quickly came up with one of the most famous results in the geometry of conics, Pascal's Theorem, which we discuss in Chapter 4. By the late 19th century projective geometry came to be seen as the most basic geometry, with Euclidean geometry as a significant but special case.

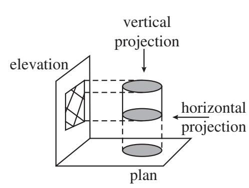
Blaise Pascal (1623–1662) was a French geometer, probabilist, physicist and philosopher.

At the start of the 19th century the world of mathematics began to change. The French Revolution saw the creation of the École Polytechnique in Paris in 1794, an entirely new kind of institution for the training of military engineers. It was staffed by mathematicians of the highest calibre, and run for many years by Gaspard Monge, an enthusiastic geometer who had invented a simple system of *descriptive geometry* for the design of forts and other military sites. Monge was one of those rare teachers who get students to see what is going on, and he inspired a generation of French geometers. The École Polytechnique, moreover, was the sole entry-point for any one seeking a career in engineering in France, and the stranglehold of the mathematicians ensured that all students received a good, rigorous education in mathematics before entering the specialist engineering schools. Thus prepared they then assisted Napoleon's armies everywhere across Europe and into Egypt.

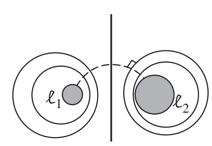
Gaspard Monge (1746–1818) was a French analyst and geometer. A strong republican and supporter of the Revolution, he was French Minister of the Navy in 1792–93, but deprived of all his honours on the restoration of the French monarchy.

One of the École's former students, Jean Victor Poncelet, was taken prisoner in 1812 in Napoleon's retreat from Moscow. He kept his spirits up during a terrible winter by reviewing what his old teacher, Monge, had taught him about descriptive geometry. This is a system of projections of a solid onto a plane – or rather two projections, one vertically and one horizontally (giving what are called to this day the *plan* and *elevation* of the solid). Poncelet realized that instead of projecting 'from infinity' so to speak, one could adapt Monge's ideas to the study of projection from a point. In this way he re-discovered Desargues' ideas of projective geometry. During his imprisonment he wrote his famous book *Traité des propriétés projectives des figures* outlining the foundations of projective geometry, which he extensively rewrote after his release in 1814 and published in 1822.

Jean Victor Poncelet (1788–1867) followed a career as a military engineer by becoming Professor of Mechanics at Metz, where he worked on the efficiency of turbines.



Around the same time that projective geometry was emerging, mathematicians began to realize that there was more to be said about circles than they had previously thought. For instance, in the study of electrostatics let ℓ_1 and ℓ_2 be two infinitely long parallel cylinders of opposite charge. Then the intersection of the surfaces of equipotential with a vertical plane is two families of circles (and a single line), and a point charge placed in the electrostatic field moves along a circular path through a specific point inside each cylinder, at right angles to circles in the families. The study of properties of such families of circles gave rise to a new geometry, called *inversive geometry*, which was able to provide particularly striking proofs of previously known results in Euclidean geometry as well as new results.



In inversive geometry mathematicians had to add a ‘point at infinity’ to the plane, and had to regard circles and straight lines as equivalent figures under the natural mappings, inversions, as these can turn circles into lines, and vice-versa. Analogously, in projective geometry mathematicians had to add a whole ‘line at infinity’ in order to simplify the geometry, and found that there were projective transformations that turned hyperbolas into ellipses, and so on. So mathematicians began to move towards thinking of geometry as the study of shapes and the transformations that preserve (at least specified properties of) those shapes.

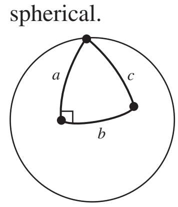
For example, there are very few theorems in Euclidean geometry that depend on the size of the figure. The ability to make scale copies without altering ‘anything important’ is basic to mathematical modelling and a familiar fact of everyday life. If we wish to restrict our attention to the transformations that preserve length, we deal with Euclidean geometry, whereas if we allow arbitrary changes of scale we deal with *similarity geometry*.

Another interesting geometry was discovered by Möbius in the 1820s, in which transformations of the plane map lines to lines, parallel lines to parallel lines, and preserve ratios of lengths along lines. He called this geometry *affine geometry* because any two figures related by such a transformation have a likeness or affinity to one another. This is the geometry appropriate, in a sense, to Monge’s descriptive geometry, and the geometry that describes the shadows of figures in sunlight.

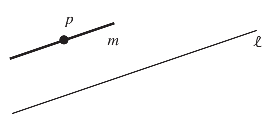
August Ferdinand Möbius (1790–1868) was a German geometer, topologist, number theorist and astronomer; he discovered the famous Möbius Strip (or Band).

Since the days of Greek mathematics, with a stimulus provided by the needs of commercial navigation, mathematicians had studied *spherical geometry* too; that is, the geometry of figures on the surface of a sphere. Here geometry is rather different from plane Euclidean geometry; for instance the area of a triangle is proportional to the amount by which its angle sum exceeds π, and there is a nice generalization of Pythagoras’ Theorem, which says that in a right-angled triangle with sides a, b and the hypotenuse c, then $\cos c = \cos a \cdot \cos b$. It turns out that there is a close connection between spherical geometry and inversive geometry.

For the surface of the Earth is very nearly spherical.



For nearly two millennia mathematicians had accepted as obvious the *Parallel Postulate* of Euclid: namely, that given any line ℓ and any point P not on ℓ, there is a unique line m in the same plane as P and ℓ which passes through P and does not meet ℓ. Indeed much effort had been put into determining whether this Postulate could be deduced from the other assumptions of Euclidean geometry. In the 1820s two young and little-known mathematicians, Bolyai in Hungary and Lobachevskii in Russia, showed that there were perfectly good so-called ‘*non-Euclidean geometries*’, namely *hyperbolic geometry* and *elliptic geometry*, that share all the initial assumptions of Euclidean geometry except the parallel postulate.



Janos Bolyai (1802–1860) was an officer in the Hungarian Army.

Nicolai Ivanovich Lobachevskii (1792–1856) was a Russian geometer who became Rector of the University of Kazan.

In hyperbolic geometry given any line ℓ and any point P not on ℓ, there are infinitely many lines in the same plane as P and ℓ which pass through P and do not meet ℓ; in elliptic geometry all lines intersect each other. However, it still makes sense in both hyperbolic and elliptic geometries to talk about the length of line segments, the distance between points, the angles between lines, and so forth. Around 1900 Poincaré did a great deal to popularise these geometries by demonstrating their applications in many surprising areas of mathematics, such as Analysis.

Jules Henri Poincaré (1854–1912) was a prolific French mathematician, physicist, astronomer and philosopher at the University of Paris.

By 1870, the situation was that there were many geometries: Euclidean, affine, projective, inversive, hyperbolic and elliptic geometries. One way mathematicians have of coping with the growth of their subject is to re-define it so that different branches of it become branches of the same subject. This was done for geometry by Klein, who developed a programme (the *Erlangen Programme*) for classifying geometries. His elegant idea was to regard a *geometry* as a space together with a group of transformations of that space; the properties of figures that are not altered by any transformation in the group are their geometrical properties.

Christian Felix Klein (1849–1925) was a German algebraist, geometer, topologist and physicist; he became a professor at the University of Erlangen at the remarkable age of 22.

For example, in two-dimensional Euclidean geometry the space is the plane and the group is the group of all length-preserving transformations of the plane (or *isometries*). In projective geometry the space is the plane enlarged (in a way we make precise in Chapter 6) by a line of extra points, and the group is the group of all continuous transformations of the space that preserve *cross-ratio*.

Klein's approach to a geometry involves three components: a set of points (the space), a set of transformations (that specify the invariant properties – for example, congruence in Euclidean geometry), and a group (that specifies how the transformations may be composed). The transformations and their group are the fundamental components of the geometry that may be applied to different spaces. A *model of a geometry* is a set which possesses all the properties of the geometry; two different models of any geometry will be isomorphic. There may be several different models of a given geometry, which have different advantages and disadvantages. Therefore, we shall use the terms 'geometry' and 'model (of a geometry)' interchangeably whenever we think that there is no risk of confusion.

For example, you will meet two models of hyperbolic geometry.

In fact as Klein was keen to stress, most geometries are examples of projective geometry with some extra conditions. For example, affine geometry emerges as the geometry obtained from projective geometry by selecting a line and considering only those transformations that map that line to itself; the line can then be thought of as lying 'at infinity' and safely ignored. The result was that Klein not only had a real insight into the nature of geometry, he could even show that projective geometry was almost the most basic geometry.

This philosophy of geometry, called the *Kleinian view of geometry*, is the one we have adopted in this book. We hope that you will enjoy this introduction to the various geometries that it contains, and go on to further study of one of the oldest, and yet most fertile, branches of mathematics.

1 Conics

The study of conics is well over 2000 years old, and has given rise to some of the most beautiful and striking results in the whole of geometry.

In Section 1.1 we outline the Greek idea of a *conic section* – that is, a conic as defined by the curve in which a double cone is intersected by a plane. We then look at some properties of circles, the simplest of the non-degenerate conics, such as the condition for two circles to be *orthogonal* and the equations of the family of all circles through two given points.

That is, they intersect at right angles.

We explain the focus–directrix definition of the parabola, ellipse and hyperbola, and study the focal-distance properties of the ellipse and hyperbola. Finally, we use the so-called *Dandelin spheres* to show that the Greek conic sections are just the same as the conics defined in terms of a focus and a directrix.

In Section 1.2 we look at tangents to conics, and the reflection properties of the parabola, ellipse and hyperbola. It turns out that these are useful in practical situations as diverse as anti-aircraft searchlights and astronomical optical telescopes! We also see how we can construct each non-degenerate conic as the ‘envelope’ of lines in a suitably-chosen family of lines.

The equations of conics are all second degree equations in x and y. In Section 1.3 we show that the converse result holds – that is, that every second degree equation in x and y represents a conic. We also find an algorithm for determining from its equation in x and y which type of non-degenerate conic a given second degree equation represents, and for finding its principal features.

The analogue in $\mathbb{R}^3$ of a plane conic in $\mathbb{R}^2$ is a *quadric surface*, specified by a suitable second degree equation in x, y and z. A well-known example of a quadric surface is the cooling tower of an electricity generating station. In Section 1.4 we find an algorithm for identifying from its equation which type of non-degenerate quadric a given second degree equation in x, y and z represents. We also discover that two of the non-degenerate quadric surfaces can be generated by two different families of straight lines, and that this feature is of practical importance.

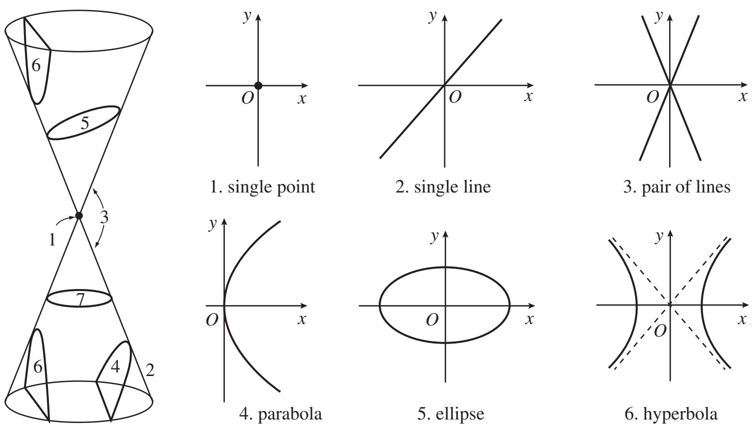
We use the notation $\mathbb{R}^2$ and $\mathbb{R}^3$ to denote 2-dimensional and 3-dimensional Euclidean space, respectively.

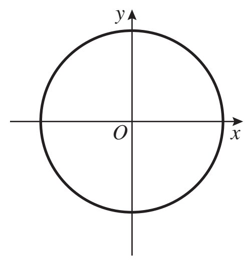
1.1 Conic Sections and Conics

1.1.1 Conic Sections

It is thought that the Greek mathematician Menaechmus discovered the conic sections around 350 BC.

Conic Section is the name given to the shapes that we obtain by taking different plane slices through a double cone. The shapes that we obtain from these cross-sections are as drawn below.





7. Circle

Notice that the circle shown in slice 7 can be regarded as a special case of an ellipse.

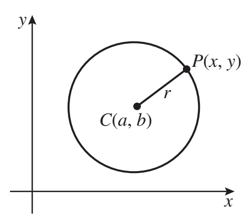
Notice, also, that the ellipse and the hyperbola both have a *centre;* that is, there is a point C such that rotation about C through an angle π is a symmetry of the conic. For example, for the ellipse and hyperbola illustrated above, the centre is in fact just the origin. On the other hand, the parabola does not have a centre.

In Subsection 1.1.5 we shall verify that the curves, the ‘conic sections’, obtained by slicing through a double cone are exactly the same curves, the ‘conics’, obtained as the locus of points in the plane whose distance from a fixed point is a constant multiple of its distance from a fixed line. As a result, we often choose not to distinguish between the terms ‘conic section’ and ‘conic’!

We use the term *non-degenerate conics* to describe those *conics* that are parabolas, ellipses or hyperbolas; and the term *degenerate conics* to describe the single point, single line and pair of lines.

In this chapter we study conics for their own interest, and we will meet them frequently throughout our study of geometry as the book progresses.

1.1.2 Circles



The first conic that we investigate is the circle. Recall that a *circle* in $\mathbb{R}^2$ is the set of points (x, y) that lie at a fixed distance, called the *radius,* from a fixed point, called the *centre* of the circle. We can use the techniques of coordinate geometry to find the equation of a circle with given centre and radius.

Let the circle have centre $C(a, b)$ and radius r. Then, if $P(x, y)$ is an arbitrary point on the circumference of the circle, the distance CP equals r. It follows from the formula for the distance between two points in the plane that

$$r^2 = (x-a)^2 + (y-b)^2. \quad (1)$$

Here we use the *Distance Formula* for the distance d between two points (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

If we now expand the brackets in equation (1) and collect the corresponding terms, we can rewrite equation (1) in the form

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0.$$

Then, if we write f for $-2a$, g for $-2b$ and h for $a^2 + b^2 - r^2$, this equation takes the form

$$x^2 + y^2 + fx + gy + h = 0. \quad (2)$$

Note here that the coefficients of x^2 and y^2 are equal.

It turns out that in many situations, however, equation (1) is more useful than equation (2) for determining the equation of a particular circle.

Theorem 1 The equation of a circle in $\mathbb{R}^2$ with centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2.$$

For example, it follows from this formula that the circle with centre $(-1, 2)$ and radius $\sqrt{3}$ has equation

$$(x+1)^2 + (y-2)^2 = \left(\sqrt{3}\right)^2;$$

this can be simplified to give

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 3,$$

or

$$x^2 + y^2 + 2x - 4y + 2 = 0.$$

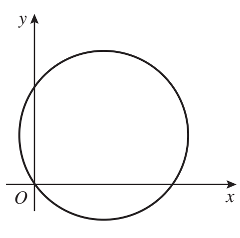
Problem 1 Determine the equation of each of the circles with the following centre and radius:

(a) centre the origin, radius 1;
(b) centre the origin, radius 4;
(c) centre (3, 4), radius 2;
(d) centre (3, 4), radius 3.

Problem 2 Determine the condition on the numbers f, g and h in the equation

$$x^2 + y^2 + fx + gy + h = 0$$

for the circle with this equation to pass through the origin.



We have seen that the equation of a circle can be written in the form

$$x^2 + y^2 + fx + gy + h = 0. \quad (2)$$

In the opposite direction, given an equation of the form (2), can we determine whether it represents a circle? If it does represent a circle, can we determine its centre and radius?

For example, consider the set of points (x, y) in the plane that satisfy the equation:

$$x^2 + y^2 - 4x + 6y + 9 = 0. \quad (3)$$

Note that in equation (3) the coefficients of x^2 and y^2 are both 1.

In order to transform equation (3) into an equation of the form (1), we use the technique called 'completing the square' — we rewrite the terms that involve only xs and the terms that involve only ys as follows:

$$x^2 - 4x = (x-2)^2 - 4,$$
$$y^2 + 6y = (y+3)^2 - 9.$$

Note that −2 is half the coefficient of x, and +3 is half the coefficient of y, in equation (3).

Substituting these expressions into equation (3), we obtain

$$(x-2)^2 + (y+3)^2 = 4.$$

We can 'read off' the centre and radius of the circle from this equation.

It follows that the equation represents a circle whose centre is (2, −3) and whose radius is 2.

In general, we can use the same method of 'completing the square' to rewrite the equation

$$x^2 + y^2 + fx + gy + h = 0$$

Here we start with the coefficients of x^2 and y^2 both equal (to 1). Otherwise the equation cannot be reformulated in the form (1).

in the form

$$\left(x + \tfrac{1}{2}f\right)^2 + \left(y + \tfrac{1}{2}g\right)^2 = \tfrac{1}{4}f^2 + \tfrac{1}{4}g^2 - h, \quad (4)$$

from which we can 'read off' the centre and radius.

Theorem 2 An equation of the form

$$x^2 + y^2 + fx + gy + h = 0$$

represents a circle with

$$\text{centre } \left(-\tfrac{1}{2}f, -\tfrac{1}{2}g\right) \text{ and radius } \sqrt{\tfrac{1}{4}f^2 + \tfrac{1}{4}g^2 - h},$$

provided that $\frac{1}{4}f^2 + \frac{1}{4}g^2 - h > 0$.

Remark

It follows from equation (4) above that if $\frac{1}{4}f^2 + \frac{1}{4}g^2 - h < 0$, then there are no points (x, y) that satisfy the equation $x^2 + y^2 + fx + gy + h = 0$; and if $\frac{1}{4}f^2 + \frac{1}{4}g^2 - h = 0$, then the given equation simply represents the single point $\left(-\frac{1}{2}f, -\frac{1}{2}g\right)$.

Problem 3 Determine the centre and radius of each of the circles given by the following equations:

(a) $x^2 + y^2 - 2x - 6y + 1 = 0$; (b) $3x^2 + 3y^2 - 12x - 48y = 0$.

Problem 4 Determine the set of points (x, y) in $\mathbb{R}^2$ that satisfies each of the following equations:

(a) $x^2 + y^2 + x + y + 1 = 0$;
(b) $x^2 + y^2 - 2x + 4y + 5 = 0$;
(c) $2x^2 + 2y^2 + x - 3y - 5 = 0$.

Orthogonal Circles

We shall sometimes be interested in whether two intersecting circles are *orthogonal*: that is, whether they meet at right angles. The following result answers this question if we know the equations of the two circles.

For example, in Chapters 5 and 6.

Theorem 3 Orthogonality Test
Two intersecting circles C_1 and C_2 with equations

$$x^2 + y^2 + f_1x + g_1y + h_1 = 0 \quad \text{and}$$
$$x^2 + y^2 + f_2x + g_2y + h_2 = 0,$$

respectively, are orthogonal if and only if

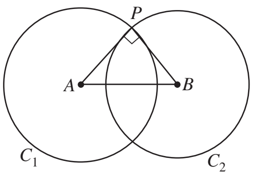
$$f_1f_2 + g_1g_2 = 2(h_1 + h_2).$$

Proof The circle C_1 has centre $A = \left(-\frac{1}{2}f_1, -\frac{1}{2}g_1\right)$ and radius $r_1 = \sqrt{\frac{1}{4}f_1^2 + \frac{1}{4}g_1^2 - h_1}$; the circle C_2 has centre $B = \left(-\frac{1}{2}f_2, -\frac{1}{2}g_2\right)$ and radius $r_2 = \sqrt{\frac{1}{4}f_2^2 + \frac{1}{2}g_2^2 - h_2}$.

You met these formulas in Theorem 2.

Let P be one of their points of intersection, and look at the triangle ΔABP. If the circles meet at right angles, then the line AP is tangential to the circle C_2, and is therefore at right angles to the line BP. So the triangle ΔABP is right-angled, and we may apply Pythagoras' Theorem to it to obtain

We use the symbol Δ to indicate a triangle.



$$AP^2 + BP^2 = AB^2. \tag{5}$$

Conversely, if equation (5) holds, then ΔABP must be a right-angled triangle and the circles must meet at right angles.

Now

$$AP^2 = r_1^2 = \tfrac{1}{4}f_1^2 + \tfrac{1}{4}g_1^2 - h_1 \quad \text{and}$$
$$BP^2 = r_2^2 = \tfrac{1}{4}f_2^2 + \tfrac{1}{4}g_2^2 - h_2.$$

Also

$$\begin{aligned} AB^2 &= \left(\tfrac{1}{2}f_1 - \tfrac{1}{2}f_2\right)^2 + \left(\tfrac{1}{2}g_1 - \tfrac{1}{2}g_2\right)^2 \\ &= \left(\tfrac{1}{4}f_1^2 - \tfrac{1}{2}f_1f_2 + \tfrac{1}{4}f_2^2\right) + \left(\tfrac{1}{4}g_1^2 - \tfrac{1}{2}g_1g_2 + \tfrac{1}{4}g_2^2\right). \end{aligned}$$

Substituting for AP^2, BP^2 and AB^2 into equation (5), and cancelling common terms, we deduce that equation (5) is equivalent to

$$-h_1 - h_2 = -\tfrac{1}{2} f_1 f_2 - \tfrac{1}{2} g_1 g_2,$$

that is,

$$f_1 f_2 + g_1 g_2 = 2(h_1 + h_2).$$

This is the required result. ∎

Problem 5 Determine which, if any, of the following pairs of intersecting circles are mutually orthogonal.

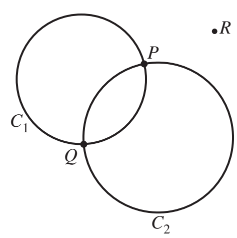
(a) $C_1 = \{(x, y) : x^2 + y^2 - 4x - 4y + 7 = 0\}$ and
$C_2 = \{(x, y) : x^2 + y^2 + 2x - 8y + 5 = 0\}$

(b) $C_1 = \{(x, y) : x^2 + y^2 + 3x - 6y + 5 = 0\}$ and
$C_2 = \{(x, y) : 3x^2 + 3y^2 + 4x + y - 15 = 0\}$.

Circles through Two Points

We shall also be interested later in the family of circles through two given points. So, let two circles C_1 and C_2 with equations

Section 5.5



$$x^2 + y^2 + f_1 x + g_1 y + h_1 = 0 \quad \text{and}$$
$$x^2 + y^2 + f_2 x + g_2 y + h_2 = 0 \qquad (6)$$

intersect at the distinct points P and Q, say. Then, if $k \neq -1$, the equation

$$x^2 + y^2 + f_1 x + g_1 y + h_1 + k(x^2 + y^2 + f_2 x + g_2 y + h_2) = 0 \qquad (7)$$

represents a circle since it is a second degree equation in x and y with equal (non-zero) coefficients of x^2 and y^2 and with no terms in xy. This circle passes through both P and Q; for the coordinates of P and Q both satisfy the equations in (6) and so must satisfy equation (7).

If $k = -1$, equation (7) is linear in x and y, and so represents a line; since P and Q both lie on it, it must be the line through P and Q.

Conversely, given any point R in the plane that does not lie on the circle C_2 we can substitute the coordinates of R into equation (7) to find the unique value of k such that the circle with equation (7) passes through R. We can think of the circle C_2 as corresponding to the case '$k = \infty$' of equation (7). For, if we rewrite equation (7) in the form

This is possible because, since R does not lie on C_2, the term in the bracket in (7) does not vanish at R.

$$\frac{1}{k}(x^2 + y^2 + f_1 x + g_1 y + h_1) + x^2 + y^2 + f_2 x + g_2 y + h_2 = 0 \qquad (8)$$

and let $k \to \infty$, then $1/k \to 0$ and equation (8) becomes the equation of C_2.

Theorem 4 Let C_1 and C_2 be circles with equations

$$x^2 + y^2 + f_1 x + g_1 y + h_1 = 0 \quad \text{and}$$
$$x^2 + y^2 + f_2 x + g_2 y + h_2 = 0$$