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The effect of points fattening in dimension three

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Abstract

There has recently been increased interest in understanding the relationship between the symbolic powers of an ideal and the geometric properties of the corresponding variety. While a number of results are available for the two-dimensional case, higher dimensions are largely unexplored. In the present paper we study a natural conjecture arising from a result by Bocci and Chiantini. As a first step toward understanding the higher-dimensional picture, we show that this conjecture is true in dimension three. Also, we provide examples showing that the hypotheses of the conjecture may not be weakened.

Dedicated to Robert Lazarsfeld on the occasion of his sixtieth birthday

1 Introduction

The study of the effect of points fattening was initiated by Bocci and Chiantini [3]. Roughly speaking, they considered the radical ideal I of a finite set Z of points in the projective plane, its second symbolic power $I^{(2)}$, and deduced from the comparison of algebraic invariants of these two ideals various geometric properties of the set Z. Along these lines, Dumnicki *et al.* [7] studied higher symbolic powers of I. Similar problems were studied in [1] in the bi-homogeneous setting of ideals defining finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

It is a natural task to try to generalize the result of Bocci and Chiantini [3, Theorem 1.1] to the higher-dimensional setting. Denoting by $\alpha(I)$ the *initial*

degree of a homogeneous ideal I, i.e., the least degree k such that $(I)_k \neq 0$, a natural generalization reads as follows:

Conjecture 1.1 *Let Z be a finite set of points in projective space $\mathbb{P}^n$ and let I be the radical ideal defining Z. If*

$$d := \alpha(I^{(n)}) = \alpha(I) + n - 1 \,, \tag{1}$$

then either

$\alpha(I) = 1$, *i.e., Z is contained in a single hyperplane H in $\mathbb{P}^n$*

or

Z consists of all intersection points (i.e., points where n hyperplanes meet) of a general configuration of d hyperplanes in $\mathbb{P}^n$, i.e., Z is a star *configuration. For any polynomial in $I^{(n)}$ of degree d, the corresponding hypersurface decomposes into d such hyperplanes.*

The term *general configuration* in the conjecture means simply that no more than n hyperplanes meet in one point. This is equivalent to the *general linear position* for points in the dual projective space corresponding to the hyperplanes in the configuration. The result of Bocci and Chiantini is the case $n = 2$ of this conjecture. As a first step toward understanding the higher-dimensional picture, we show in the present paper:

Theorem 1.2 *The conjecture is true for $n = 3$.*

The assumption on the ideal I in the theorem amounts to the two equalities

$$\alpha(I^{(2)}) = \alpha(I) + 1$$
$$\alpha(I^{(3)}) = \alpha(I^{(2)}) + 1$$

and one might be tempted to relax the assumptions to only one of them. In Section 6 we provide examples showing, however, that neither is sufficient by itself to reach the conclusion of the theorem.

Star configurations are interesting objects of study in their own right. They are defined in [10] as unions of linear subspaces of fixed codimension c in projective space $\mathbb{P}^n$ that result as subspaces where exactly c of a fixed finite set of general hyperplanes in $\mathbb{P}^n$ intersect. The case described in Conjecture 1.1 corresponds thus to the $c = n$ situation. It is natural to wonder if the following further generalization of Conjecture 1.1 might be true: If Z is a finite collection of linear subspaces of codimension $c \leqslant n$ in $\mathbb{P}^n$ with the radical ideal I and such that

$$d = \alpha(I^{(c)}) = \alpha(I) + c - 1 \,,$$

then Z is either contained in a hyperplane or forms a star configuration of codimension c subspaces. Janssen's recent preprint [12] deals with lines in $\mathbb{P}^3$ and shows that such a simple generalization would be too naive. Nevertheless we expect that there are some undiscovered patterns lurking beneath, and we hope to come back to this subject in the near future.

Throughout this paper we work over the complex numbers, and we use standard notation in algebraic geometry as in [13].

2 Initial degrees of symbolic powers

Definition 2.1 (Symbolic power)   Let I be a homogeneous ideal in the polynomial ring $R = \mathbb{C}[\mathbb{P}^n]$. For a positive integer k, the ideal

$$I^{(k)} = R \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} I^k R_{\mathfrak{p}} \right),$$

where the intersection is taken in the field of fractions of R, is the kth *symbolic power* of I.

Definition 2.2 (Differential power)   Let I be a radical homogeneous ideal and let $V \subset \mathbb{P}^n$ be the corresponding subvariety. For a positive integer k, the ideal

$$I^{\langle k \rangle} = \bigcap_{P \in V} \mathfrak{m}_P^k,$$

where $\mathfrak{m}_P$ denotes the maximal ideal defining the point $P \in \mathbb{P}^n$, is the kth *differential power* of I.

In other words, the kth differential power of an ideal consists of all homogeneous polynomials vanishing to order at least k along the underlying variety. For radical ideals these two concepts fall together due to a result of Nagata and Zariski, see [8, Theorem 3.14] for prime ideals and [14, Corollary 2.9] for radical ideals:

Theorem 2.3 (Nagata–Zariski)   *If I is a radical ideal in a polynomial ring over an algebraically closed field, then*

$$I^{\langle k \rangle} = I^{(k)}$$

for all $k \geqslant 1$.

We will make use of the following observation on symbolic powers:

Lemma 2.4 *Let I be an arbitrary radical homogeneous ideal. Then we have the inequality*

$$\alpha(I^{(k+1)}) - \alpha(I^{(k)}) \geqslant 1$$

for all $k \geqslant 1$.

Proof Let Z be the subscheme of $\mathbb{P}^n$ defined by I. The claim of the lemma follows immediately from the interpretation of symbolic powers as differential powers, Theorem 2.3, and the observation that if a polynomial f vanishes along Z to order at least $k + 1$, then any of its partial derivatives vanishes along Z to order at least k. □

3 The $\mathbb{P}^2$ case revisited

As a warm-up, we give here a new proof of the result of Bocci and Chiantini. This proof has the advantage that it does not make use of the Plücker formulas.

Theorem 3.1 (Bocci–Chiantini) *Let Z be a finite set of points in the projective plane $\mathbb{P}^2$ and let I be its radical ideal. If*

$$d = \alpha(I^{(2)}) = \alpha(I) + 1 \,,$$

then either Z consists of collinear points or Z is the set of all intersection points of a general configuration of d lines in $\mathbb{P}^2$.

Proof If $d = 2$, then we are done. So we assume $d \geqslant 3$.

By Lemma 4.2 below we may assume that Z consists of exactly $\binom{d}{2}$ points. Let X_2 be a divisor of degree d that is singular in all points of Z. Let P be one of the points in Z. Then there exists a divisor W_P of degree $d - 2$ vanishing at all points in $Z \setminus \{P\}$ (and not vanishing at P).

We claim that W_P is contained in X_2. To see this, we begin by showing that they must have a common component. Indeed, this follows from Bézout's theorem, since otherwise we would get

$$d(d - 2) = X_2 \cdot W_P \geqslant 2\left(\binom{d}{2} - 1\right) = d(d - 1) - 2 \,,$$

which is equivalent to $d \leqslant 2$ and contradicts our initial assumption in this proof.

Now let Γ be the greatest common divisor of X_2 and W_P, and let e be the degree of the divisor $W'_P = W_P - \Gamma$ (so that $\deg(\Gamma) = d - 2 - e$). There must be at least $\binom{e+2}{2} - 1$ points from $Z \setminus \{P\}$ on W'_P (otherwise there would be a pencil of such divisors W'_P and one could choose an element in this pencil

passing through P, but then $W_P' + \Gamma$ would be an element of degree $d - 2$ in I contradicting the assumption on $\alpha(I)$).

Then we intersect again $X_2 - \Gamma$ with W_P' and obtain

$$(e + 2)e = (X_2 - \Gamma) \cdot W_P' \geqslant 2 \left(\binom{e + 2}{2} - 1 \right) = (e + 2)(e + 1) - 2 \, ,$$

which gives $e = 0$.

It follows that $X_2 - W_P$ is a divisor of degree 2 with a double point at P. Hence, X_2 contains two lines intersecting in P. This holds for every point $P \in Z$. Since X_2 can contain at most d lines, we see that this is only possible if Z consists of the intersection points of a general configuration of d lines. $\qquad\square$

4 A reduction result

We begin with a lemma concerning star configurations of points. We include the proof, since we were not able to trace it in the literature.

Lemma 4.1 *Let Z be a star configuration of points defined by hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^n$. For $d \geqslant n + 1$ the union*

$$H_1 \cup \ldots \cup H_d$$

is the only hypersurface F of degree d with the property

$$\mathrm{mult}_P F \geqslant n \ \text{for all} \ \ P \in Z. \tag{2}$$

Proof We proceed by induction on the dimension $n \geqslant 2$. The initial case of $\mathbb{P}^2$ is dealt with simply by a Bézout-type argument. Indeed, assuming that there exists a curve F of degree d passing through all points in Z with multiplicity $\geqslant 2$ and taking a configuration line H_i, Bézout's theorem implies that H_i is a component of F. Since this holds for all lines in the configuration and $\deg(F) = d$, we are done.

For the induction step we assume that the lemma holds for dimension $n - 1$ and all $d \geqslant n$. We want to conclude that it holds for $\mathbb{P}^n$ and all $d \geqslant n + 1$. Of course we may assume that $n \geqslant 3$.

To this end let F by a hypersurface of degree d in $\mathbb{P}^n$ satisfying (2). Suppose that there exists a hyperplane H among $H_1, \ldots, H_d$, which is not a component of F. Then the restriction $G = F \cap H$ is a hypersurface of degree d in $H \simeq \mathbb{P}^{n-1}$ with $\mathrm{mult}_P G \geqslant n$ for all $P \in Z_H = Z \cap H$. Note that Z_H is itself a star configuration of points in H, defined by hyperplanes obtained as intersections $H_i \cap H$. So it is a star configuration of $d - 1$ hyperplanes in $\mathbb{P}^{n-1}$. The polar

system of G (i.e., the linear system defined by all first-order derivatives of the equation of G) is of dimension $n - 1 \geqslant 2$, and every element K in this system satisfies

$$\text{mult}_P K \geqslant n - 1 \quad \text{for all } P \in Z_H.$$

This contradicts the induction assumption. $\square$

The following lemma allows us to assume $\#Z = \binom{d}{n}$ when proving the theorem (or when working on the conjecture).

Lemma 4.2   *Suppose that the set $Z \subset \mathbb{P}^n$ satisfies the assumptions of Conjecture 1.1 and that $\alpha(I) \geqslant 2$. Then there is a subset $W \subset Z$ with the following properties:*

(i)   W is of cardinality $\binom{d}{n}$.
(ii) For the ideal J of W we have $\alpha(J^{(k)}) = \alpha(I^{(k)})$ for $k = 1, \ldots, n$.
(iii) If W is a star configuration, then $W = Z$.

Proof   To begin with, note that the equality in (1) together with Lemma 2.4 implies $\alpha(I^{(k)}) = d - n + k$ for $k = 1, \ldots, n$.

The assumption $\alpha(I) \geqslant 2$ implies $d \geqslant n + 1$. Since there is no form of degree $\leqslant d - n$ vanishing along Z, there must be at least

$$s := \binom{d}{n} \tag{3}$$

points in Z.

Now we choose exactly s points $P_1, \ldots, P_s$ from Z that impose independent conditions on forms of degree $d - n$. (This can be done, since vanishing at each point in Z gives a linear equation on the coefficients of a form of degree $d - n$, so that we obtain a system of $\#Z$ linear equations of rank $s = \binom{d}{n}$ (which is the maximal possible rank). Then we choose a subsystem of s equations with maximal rank.) Let $W := \{P_1, \ldots, P_s\}$ and let J be the radical ideal of W. Since $W \subset Z$, we certainly have

$$\alpha(J^{(k)}) \leqslant \alpha(I^{(k)})$$

for all $k \geqslant 0$. On the other hand, we have

$$\alpha(J) = d - n + 1 = \alpha(I)$$

by the selection of W. Lemma 2.4 then implies that in fact

$$\alpha(J^{(k)}) = \alpha(I^{(k)})$$

for $k = 1, \ldots, n$. This shows that conditions (i) and (ii) are satisfied.

As for (iii): Suppose that W is a star configuration. By (ii) we have $\alpha(I^{(n)}) = \alpha(J^{(n)})$, hence it follows from Lemma 4.1 that $W = Z$. □

Further, we need the following elementary lemma on hypersurfaces that are obtained by taking derivatives.

Lemma 4.3 *Let $X \subset \mathbb{P}^n$ be a hypersurface defined by a polynomial f of degree d with a point P of multiplicity m such that $2 \leqslant m < d$. Then there exists a direction v such that the hypersurface Λ defined by the directional derivative of f in direction v has multiplicity m at P.*

Proof After a projective change of coordinates, we may assume $P = (1 : 0 : \ldots : 0)$. Then we can write

$$
\begin{aligned}
f(x_0 : x_1 : \ldots : x_n) = {} & x_0^{d-m} g_m(x_1 : \ldots : x_n) \\
& + x_0^{d-m-1} g_{m+1}(x_1 : \ldots : x_n) \\
& + \ldots \\
& + g_d(x_1 : \ldots : x_n)
\end{aligned}
$$

with homogeneous polynomials g_i of degree i for $i = m, \ldots, d$. Since $d > m$, the divisor defined by $\frac{\partial f}{\partial x_0} = 0$ has multiplicity m at P. □

5 Dimension three

In this section we give the

Proof of Theorem 1.2 We proceed by induction on d. For $d \leqslant 3$ the statement of the theorem is trivially satisfied, so we assume $d \geqslant 4$. By Lemma 4.2 we may assume that Z is of cardinality $\binom{d}{3}$. Let $X_3 \subset \mathbb{P}^3$ be the divisor defined by a polynomial of degree d in $I^{(3)}$. We assert that

$$X_3 \text{ is reducible.} \tag{$*$}$$

To see this, we first note that thanks to $m = 3 < 4 \leqslant d$ there is, by Lemma 4.3, for any $P \in Z$ a directional derivative surface Λ_P of degree $d-1$ with multiplicity at least 3 at P. Arguing by contradiction, we assume that X_3 is irreducible, which implies that X_3 and Λ_P intersect properly, i.e., in a curve. Adapting the proof of [11, Proposition 3.1] to dimension 3, we see that the linear system of forms of degree $d - 2$ vanishing along Z has only Z as its base locus. (This is due to the fact that the regularity of I is $d - 2$.) We can therefore choose an element Y in this system that does not contain any component of the intersection

curve of X_3 and Λ_P. Then the three surfaces X_3, Λ_P and Y intersect in points only, and we can apply Bézout's theorem to get

$$d(d-1)(d-2) = X \cdot \Lambda_P \cdot Y \geqslant 6\left(\binom{d}{3} - 1\right) + 9 \,.$$

But this implies $0 \geqslant 3$, a contradiction. So (*) is established.

Now let Γ be an irreducible component of X_3 of smallest degree. Set $\gamma = \deg \Gamma$ and $X'_3 = X_3 - \Gamma$. Our aim is to apply induction on X'_3. To this end we consider the set

$$Z' = Z \setminus \Gamma \,.$$

It is non-empty, as otherwise Z would be contained in Γ and then $\alpha(I)$ would be less than $d-2$. Indeed, we would have $\alpha(I) \leqslant \gamma \leqslant \left\lfloor \frac{d}{2} \right\rfloor$, which is less than $d-2$ if $d \geqslant 5$; and if $d = 4$, then $\gamma = 2$, so X_3 consists of two quadrics, which implies that it can have only two triple points – but then $\alpha(I) = 1$.

As Z' is non-empty, there is in particular a triple point on X'_3, and hence $d - \gamma = \deg X'_3 \geqslant 3$.

We claim that

$$\alpha(I_{Z'}) \geqslant d - \gamma - 2 \,. \tag{4}$$

In fact, there is otherwise a surface S of degree $d - \gamma - 3$ passing through Z', and then $S + \Gamma$ is a divisor of degree $d-3$ passing through Z, which contradicts the assumption $\alpha(I) = d - 2$.

Next, note that

$$\alpha(I_{Z'}^{(3)}) \leqslant \deg X'_3 = d - \gamma \,. \tag{5}$$

In fact, as Γ does not pass through any of the points of Z', we know that X'_3 has multiplicity at least 3 on Z'.

By Lemma 2.4, we obtain from (5) the inequality

$$\alpha(I_{Z'}) \leqslant d - \gamma - 2 \tag{6}$$

and this shows with (4) that equality holds in (6). From (5) we see, again with Lemma 2.4, that equality holds in (5) as well. We have thus established that the assumptions of the theorem are satisfied for the set Z'. By induction we conclude therefore that Z' is a star configuration and that X'_3 decomposes into planes, or Z' is contained in a hyperplane and then the support of X'_3 is that hyperplane. As Γ was chosen of minimal degree, it must be a plane as well, and hence X_3 decomposes entirely into planes. We can then run the above induction argument for *any* plane component Π of X_3 to see that the surface $X_3 - \Pi$ yields a star configuration. This shows immediately that there are no triple

intersection lines among these planes and we conclude by just counting points
with multiplicity at least 3 (in fact, exactly 3) that Z is a star configuration. $\square$

6 Further results and examples

Recall that the *Waldschmidt constant* of a homogeneous ideal $I \subset \mathbb{P}^n$ is the
asymptotic counterpart of the initial degree, defined as

$$\widehat{\alpha}(I) = \lim_{k \to \infty} \frac{\alpha(I^{(k)})}{k} = \inf_{k \geqslant 1} \frac{\alpha(I^{(k)})}{k} .$$

This invariant is indeed well defined, see [4] for this fact and some basic prop-
erties of $\widehat{\alpha}$. The Waldschmidt constants are interesting invariants that were
recently rediscovered and studied by Bocci and Harbourne, see e.g. [5]. While
Harbourne introduced the notation $\gamma(I)$, we propose here the notation $\widehat{\alpha}(I)$, as
the Waldschmidt constant *is* the asymptotic version of the initial degree $\alpha(I)$,
and the notation is then consistent with [9].

We state now a corollary of Theorem 1.2 dealing with the case where there
is just one more α-jump by 1.

Corollary 6.1   *Let Z be a finite set of points in projective three-space $\mathbb{P}^3$ and
let I be the radical ideal defining Z. If*

$$d := \alpha(I^{(4)}) = \alpha(I) + 3 , \tag{7}$$

then $\alpha(I) = 1$, i.e., Z is contained in a single plane in $\mathbb{P}^3$.

Proof   Using Theorem 1.2 we need to exclude the possibility that Z forms a
star configuration. To this end we apply in the case $n = 3$ the inequality

$$\widehat{\alpha}(I) \geqslant \frac{\alpha(I) + n - 1}{n} , \tag{8}$$

which was proved by Demailly [6, Proposition 6]. The assumptions of his
result are satisfied by our Theorem 1.4. Combining (8) with the fact that
$\widehat{\alpha}(I) \leqslant \frac{\alpha(I^{(n+1)})}{n+1}$ yields in our situation

$$\frac{\alpha(I) + n - 1}{n} \leqslant \frac{\alpha(I) + n}{n + 1} ,$$

which gives immediately $\alpha(I) = 1$. $\square$

Remark 6.2 (Waldschmidt constant of star configuration)   Note that there is
equality in (8) for star configurations of points by [4, Proof of Theorem 2.4.3].

Remark 6.3   If Conjecture 1.1 holds for any n, then the proof of the above
corollary shows that if $\alpha(I^{(n+1)}) = \alpha(I) + n$, then $\alpha(I) = 1$.

We next provide examples showing that a single α-jump by 1 is not sufficient to reach the conclusion of Theorem 1.2.

Example 6.4 (Kummer surface)   In this example we show that in general the assumption

$$\alpha(I^{(2)}) = \alpha(I) + 1 \tag{9}$$

for an ideal I of a set of points Z in $\mathbb{P}^3$ is not sufficient to conclude that the points in Z are coplanar or form a star configuration.

To this end, let $X \subset \mathbb{P}^3$ be the classical Kummer surface associated with an irreducible principally polarized abelian surface, and let Z be the set of the 16 double points on X. It is well known (see e.g. [2, Section 10.2]) that these 16 points form a 16_6 configuration, i.e., there are 16 planes Π_i in $\mathbb{P}^3$ such that each plane Π_i contains exactly 6 double points of X (and exactly 6 planes pass through every point in Z). We claim that

$$\alpha(I) = 3 \quad \text{and} \quad \alpha(I^{(2)}) = 4 \,,$$

where I is the radical ideal of Z.

Granting this for a moment, we see immediately that the points in Z are neither coplanar nor form a star configuration, whereas the assumption in (9) is satisfied.

Turning to the proof, assume that there exists a surface S defined by an element of degree 3 in $I^{(3)}$. Let Π be one of the 16 planes Π_i. Then

$$1 \cdot 3 \cdot 4 = \Pi \cdot S \cdot X \geqslant 6 \cdot 1 \cdot 2 \cdot 2$$

implies that Π is a component of S. As the same argument works for all 16 planes, we get a contradiction. Hence $\alpha(I^{(2)}) = 4$.

A similar argument excludes the possibility that Z is contained in a quadric. We leave the details to the reader.

The following simpler example exhibiting the same phenomenon has been suggested by the referee.

Example 6.5   Let L_1, L_2, L_3 be mutually distinct and not coplanar lines in $\mathbb{P}^3$ intersecting at a point P. Let $A, B \in L_1$, $C, D \in L_2$ and $E \in L_3$ be points on these lines different from their intersection point P. Obviously the set $Z = \{A, B, C, D, E\}$ is not a star configuration. Let I be the radical ideal of Z. Then it is elementary to check that

$$\alpha(I) = 2 \ \text{ and } \ \alpha(I^{(2)}) = 3.$$

Note that 5 is the minimal number of non-coplanar points that can give $\alpha(I^{(2)}) = \alpha(I) + 1$ and not form a star configuration.