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CALCULUS
OF
VARIATIONS

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TO

MARION

IN

REMEMBRANCE

PREFACE.

The subject, commonly called the Calculus of Variations, has attracted a rather fickle attention at more or less isolated intervals in its growth. Its progress has been neither steady nor consecutive. From some cause, in its nature, or in its incompleteness, or in its presentation, it has not secured an abiding interest.

Not infrequently, investigators have been concerned with applications of the Calculus and, for their purpose, have been known to use fragmentary results.

Thus, in the theory of the potential, Dirichlet's Principle has been invoked. In instances when regard has been paid to the establishment of the Principle beyond an assumption of its intuitive truth, only the initial test belonging to weak conditions has been imposed; and a general inference has been drawn, which was not justified by that test alone.

Again, the Principle of Least Action has been made the support, and sometimes the occasional basis, of theoretical explanations of the physics of the universe: though it should be added that the introduction of kinetic foci in dynamics is the equivalent of another necessary canonical test. Even so, all the recognised tests have assumed that variations in natural phenomena must be gently regular. Variations which, remaining small and continuous in their magnitude, change in a violently regular or irregular manner within a very restricted range, have usually been ignored. yet the theory of small vibrations wields a far-flung domination.

In Newton's problem of the Solid of Least Resistance, the formal solution satisfies all the customary tests which arise through variations of the gently regular type. Still, more than a century ago, Legendre proved that the solution is mathematically unsatisfactory, though its neglect by engineers is not due solely to mathematical deficiencies.

The significance of the investigations, due to Weierstrass, is not always recognised; but their importance need not be emphasised, as though complete finality has been attained. The results, usually associated with his name, relate to only the simplest class among the problems which present themselves and which require no more than the simplest form of his specially devised analysis. There is ample scope for further research by his method, in extension of the range of its application.

The present volume attempts a systematic exposition of the subject by what, in the main, is a uniform composite process. Though it does not purport to be a history, the gradual historical growth of the successive tests has governed the arrangement. A fundamental (yet quite elementary) simplification, derived from the Weierstrass method, has been used from the beginning, even to obtain the results originally due to the founders of the subject. These limited results maintain their standing, because they provide tests which must be satisfied in simple forms of enquiry, and because they remain significant even when they are merged in the wider results obtained by the more general method of Weierstrass.

Moreover, the volume has no pretensions to an encyclopædic range. Processes and investigations, however useful in the exploration of other regions, are omitted unless they fall into the course of exposition adopted. So far as I am aware, much of its material is novel. Two sources, more than others, have been useful to me. The first of them is the Moigno-Lindelöf volume *Calcul des Variations*, published in 1861; except for the Sarrus formalities, it seems to me an admirable exposition of the older range of investigation. The other source is to be found in such access to the work of Weierstrass as has been possible. Before the year 1895, I had read a manuscript copy of notes of lectures by

Weierstrass on his treatment of single integrals of the first order, including the associated isoperimetrical problems; for the loan of the volume from their College Library, I remain indebted to the authorities of St John's College, Cambridge. Since that date, Professor Harris Hancock has published (1903) his volume, based on similar notes and on lectures by Schwarz. Unfortunately, a general expectation, that an authoritative edition of the Weierstrass lectures would be published, has not yet been realised.

Beyond the sources just mentioned and such other sources as are quoted in the text, my work is independent. Some mathematicians may wish that the exposition had been differently balanced. Some will feel regret, and may award blame, for the omission of the work of writers such as Clebsch and Hilbert—an omission not due to lack of appreciation of their researches. Whatever its merits or its demerits, the presentation is that which has appealed to me, as leading most directly to a comprehension of the subject.

An abstract of the contents of the book may be useful, as an indication of its scope.

In the first chapter, the simplest form of integral is discussed. It involves only one dependent variable, together with the first derivative. The method adopted is, in substance, the older method for restricted variations; and the results obtained, including Jacobi's test which limits the extent of the range of the integration, are typical of those that persist in all subsequent investigations, though they do not constitute the aggregate of tests of a general character. The second chapter deals with the same type of integral by the method of Weierstrass, which makes both the dependent variable and the independent variable in the older process to be functions of a new independent variable, usually selected so as not to be intrinsic to the problem; thus simultaneous independent variations can then be imposed from the beginning upon both the variables which occur. It is found that, for gently regular variations, no new tests emerge from the use of the Weierstrass method,—a conclusion not unimportant in itself—though the formal expression of the tests is modified. In the third chapter, both methods are applied to integrals, which still involve only a single original dependent variable and now include derivatives of the second order as well as those of the first order. Of the analytical material in these three chapters, convenient geometrical illustration is provided by plane curves.

The next three chapters are devoted to the discussion, by both methods, of single integrals which involve two dependent variables and one independent variable in their initial form, together with derivatives of the first order, and (less generally) of the second order, though the analytical development in the latter case is not carried so far as in the former. The increase in the number of variables does not lead to an increase in the number of significant tests, though (as is almost to be expected) the expression of the several tests tends to become more complicated. For the material in these chapters, convenient geometrical illustration is provided by skew curves.

The seventh chapter introduces the essential advance made by the Weierstrass method, through the emergence of a new additional test. The advance comes through the consideration of variations which are not restricted to be of a gently regular type. The variations are naturally required to be continuous and, as maxima and minima are being considered, they are required to be small in magnitude; but, within that small range, they are permitted to vary even abruptly, as violently as continuous curves representing rapid small oscillations or even as continuous serrated curves. Many such variations can be compounded from rudimentary variations of a selected type; and the use of the latter variation leads to the construction of a new test which, necessarily satisfied for the most elementary form, is cumulative in its effect for the composite form. This Weierstrass test is applied to single integrals which, of course, involve only ordinary derivatives. In the case of the Solid of Least Resistance, it is shewn that the solution, satisfactory under the tests associated with the gently regular type of variation, does not obey the further test associated with the strong variation, and therefore does not supply a minimum. It appears also that the Principle of Least Action does not supply a minimum: the demands of the tests, arising out of gentle variations, are satisfied; but the demand of the Weierstrass test, arising out of strong variations, is not satisfied.

The eighth chapter is devoted to the consideration of simpler problems of relative maxima and minima—the isoperimetrical problems of even ancient interest. In particular, those problems are discussed, in which the requirement of a maximum or of a minimum is obliged to fulfil the condition of allowing a coexistent related integral to maintain an assigned value. Other types of relative problems—in which, for example, persistent relations hold among the variables—are considered, though only briefly, partly because the first stage in their treatment is to be found in treatises and memoirs easily accessible.

The ninth chapter deals with double integrals which, in their initial postulation, involve one dependent variable and its two first derivatives. The concurrent geometrical illustration is, of course, provided by surfaces in ordinary space. Both the older method and the later method are used for the discussion. The treatment of the most interesting of all problems of this kind—minimal surfaces—is simplified when the Weierstrass method is used from the beginning. Schwarz's theorem, which secures the determination of a minimal surface by initially assigned conditions, has been extended so as to obtain an analytical expression of the Jacobi test in limitation of the range. The tenth chapter is devoted to two issues: one, the construction of the Weierstrass test for double integrals and a proof that it is satisfied by minimal surfaces: the other, the simplest type of isoperimetrical problem. The eleventh chapter is concerned with double integrals which involve the partial derivatives of the second order; but there is no attempt at a full discussion, mainly because, after the application of even the simpler tests, the analysis becomes unwieldy and the developments demand the differential geometry of the curvature of surfaces.

A final chapter is devoted to triple integrals, involving the first derivatives of a single dependent variable. The convenient geometrical illustration is provided by the consideration of volumes in quadruple space. Only a slight use is made of the mathematical notions of such space; and, because the geometrical considerations are mainly concerned with volumes, a three-fold amplitude finds, for most purposes, a working representation in the ordinary space of experience. The analysis, which is requisite for the full application of the Weierstrass method to triple integrals, soon becomes laboured; it is here developed only so far as to construct the necessary tests which shew that, owing to failure under the Weierstrass test, Dirichlet's Principle is not valid.

Before parting from the volume, I would thank Professor H. F. Baker, for his kindness in reading the earliest sheets of the volume. Above all, I must mention the Staff of the University Press, Cambridge. Their steady and unfailing co-operation has been my mainstay during the printing of the book. Now that my task is ended, I tender my grateful thanks to all of them who have shared our joint labour.

A. R. FORSYTH.

31 *December* 1926

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