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INTRODUCTION.

General range of the subject.

1. The range of Mathematical Analysis, usually known as the Calculus of Variations, deals with one of the earliest problems of ordinary experience. The requirement was, and is, to obtain the most profitable result from imperfectly postulated data; and the data may possibly be subjected to conditions, which likewise are imperfectly postulated. When data and conditions are expressed in analytical form, the necessary mathematical calculations cannot be effected directly, because of some essential deficiency in the information. The gap has to be filled before the resolution of the problem is attained; and the process of supplying the lacking information is indirect, as compared with the regular methods of calculation. It consists of the construction of tests, which are the mathematical expression of general conditions; it is composed of various gradual stages, sometimes independent of one another; and the ensuing requirements are combined into an aggregate which is adequate for the purpose. Usually, the predominating interest lies in the qualitative results that are constructed. Not infrequently, the subsequent quantitative calculations are ignored; they involve processes which belong to an elementary range, that is unconcerned with the mode or modes of obtaining the information lacking in the initial stage of the original statement.

Early beginnings.

2. To the ancient Greeks, with their wonderful geometry, the problem came not infrequently in a practical form: how to secure the greatest amount of land, which could be enclosed within a boundary capable of being ploughed as a contour furrow in a given time: or, geometrically expressed, how to find the shape of the closed curve, which shall enclose the largest area within a perimeter of assigned length. The Greeks—perhaps not the earliest race, in spite of their scientific achievements, and certainly not the only race, desirous of saving the most and wasting the least out of opportunities subject to assigned limitations—obtained a number of results, partly by what may be called inspired guessing or intuition; partly by assumptions as to harmonies, and perfection, in curves; partly by experiments. Though the results would not be regarded by later mathematical precisians as having been established with complete rigour, they often were sufficiently satisfactory for working purposes.

Little substantial progress beyond the old attainments was made down through the middle ages. In the East, and among the Arabs in the South West of Europe, the old mathematical subjects were not merely maintained,

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but even flourished, within the range of the old methods; which came to be processes of record and description, mostly of an arithmetical and a geometrical type. Indeed, algebra was the main subject where progress was noted and notable, though often attained in geometrical guise: even the beginning of Descartes' *Geometry* is occupied with the geometrical constructions devised for the solution of quadratic equations. Real progress began with the new world of science, which was created in the seventeenth century in Western Europe—not least with the new mechanics and new astronomy, under Galileo, Kepler, and Newton. It soon appeared that the old geometrical methods, even in their new amplifications, must merge into an entirely new method which, in character, was analytic and not synthetic. Thenceforward, what now is called Analysis came to be the main instrument in vast fields of mathematical research. The infinitesimal calculus was developed: its historical origins are marked by the morbid hostility between the partisans of Newton and the partisans of Leibnitz, which raged for many a year to the detriment of achievement in recognised knowledge. Old problems were re-stated; and they were solved once more, not infrequently with new additions and unsuspected limitations. New investigations, hitherto belonging to the region of fancy, were brought within the range of practice. In particular, during the late seventeenth century and through the eighteenth century, problems were propounded by individuals as public challenges—a habit that has survived among Academies in the present day when proceeding to the award of their prizes. The actual propounding of the problem was frequently an intimation that the challenger had achieved the solution. Specially in mathematics, this was a form of initial publication. The Bernoulli family, in its successive generations, was conspicuous for contributions of this type. Others, less publicly aggressive in tone, were content to achieve a result and to be satisfied with the knowledge thus attained, without challenge or publication in any similar form. Of such men, perhaps Newton is the most conspicuous instance.

Newton, John Bernoulli.

3. The gradual development of the infinitesimal calculus led to the formulation and the solution of new problems; and the new inquiries were not less frequent in problems, the aspect of which was mathematical rather than astronomical or mechanical or physical. Particularly within the range of problems connected with maximum attainment or minimum reduction, the new calculus proved effective; usually, the data were sufficient to allow a direct attack to be made upon the problem. But, soon, the problems sometimes became of a subtler indirect nature: the very character, not merely the magnitude, of the unknown quantities was the essence of the problem; and even behind this difficulty lay processes that could not be effected.

Such difficulties arose most directly when the maximum or the minimum was to be possessed by a quantity which, however veiled in expression at the

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time, can be formulated as an integral, definite or indefinite. The quadrature —the actual evaluation—of the integral could not be made, because it would contain unknown quantities. On the one hand, the quest of these quantities was the main problem; on the other hand, progress towards solution of the problem was barred, so far as the old methods were concerned, precisely because the quantities were unknown. Among such preliminary and isolated problems, two (among others) survive in interest to the present day.

One is Newton's problem of the determination of the shape of the solid of revolution which shall meet with the least resistance to its motion through a fluid, on the assumption of a law of resistance conforming roughly to observation. Newton's solution was not satisfactory so far as practice is concerned, even if the law of resistance be regarded as adequate; and Weierstrass's analysis provided, more than half a century ago, a reason why the Newtonian solution is not satisfactory, even from a theoretical point of view. This general problem has come into more importance in recent times, owing to the developments of submarines and air-craft; and the necessary association of new knowledge, derived from successive physical and mechanical experiments, has added grave complications to the mathematics of the general problem.

A second problem, of historical importance and still of interest, is that of the Brachistochrone, associated (1696) with the name of John Bernoulli. In its simplest form, it requires the determination of the curve joining two given points such that the time of passage of a given mass from one point to the other along the curve (as in a smooth groove), under the influence of gravity alone and subject to no retarding force, is a minimum, that is, less than the time of passage in like circumstances along any other curve joining the two points. The original solution, which requires the curve to be a cycloidal arc, is accurate as regards the quality of the characteristic curve; later investigations have added limiting specifications on the range of the curve. And, naturally, analogous problems have been propounded and solved, when the motion is due to external forces other than gravity, and when retarding forces can come into play.

Without multiplying the citation of instances unduly, it may suffice to mention the mathematical-physical conception denoted by the word Action. Philosophers have been fain to deduce the mechanical movements of a dynamical configuration, even much of the physics of the universe, from the single property, that the Action between any two states of the configuration in continuous change is a minimum; and the property has been elevated in postulation to the Principle of Least Action.

Euler and Lagrange.

4. The mathematical development of that section of Analysis, which has to be considered here, has been fitful, sporadic, slow; and, as so often is the case, not entirely free from controversy. Amid many names that now

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belong to the history of mathematical science, four stand out, because of the significance of their contributions towards the attainment of the first stage in the systematic construction of the calculus.

The earliest name is that of Euler, fruitful in that branch (as in all branches) of Analysis in his day. He discovered* the characteristic differential equation which, as a necessary requisite, must be satisfied. It was obtained by considering the increment of the integral which arose through the variation of a rudimentary arc. In the first stage, Euler stayed his investigations at this mathematical result; it secured a stationary quality, as a preliminary common to a maximum or a minimum property of the magnitude under discussion.

Lagrange† discussed the problem by regarding variations as active through the range of integration, not restricted to any rudimentary portion of that range. Indeed, he made a new foundation of the subject, entirely analytical in character as was his mathematical wont. To him is due the introduction of the symbol δ, which discharged useful duty through successive generations of mathematicians: though, as can happen from lack of definite and precise explanation, initially unexpressed, inferences by later investigators were sometimes drawn, more comprehensive in statement than could be justified. He also extended his analysis so that it could be applied to problems involving two independent variables; and, in particular, his researches constitute a foundation of the theory of minimal surfaces‡.

Subsequently, Euler resumed his consideration of the subject, on the basis of Lagrange's work. His completed account§ is based upon the geometrical representation of the analysis, in which he regarded δy as a change in the position of a point on an original curve, along the ordinate, to a contiguous point on the varied curve.

There, for a period, advance in the progress of the subject was arrested. Detailed amplifications within the range were forthcoming, but always in the field of work already achieved by Euler and Lagrange, whose names dominate the first stage in the systematic calculus.

Legendre.

5. The results obtained by Euler and Lagrange were of the nature of necessary qualifying tests. The discrimination among merely stationary values, as between a maximum or a minimum, was left to intuitive considerations, foreign to the analysis.

* *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti* (Lausanne et Geneva, 1744).

† *Miscellanea Taurinensia*, t. ii (1760–1), pp. 173—195; *ib.*, t. iv (1766–9), pp. 163—187: *Œuvres*, I, pp. 335—362, II, pp. 37—63.

‡ For reference, see the preliminary note to Chapter IX.

§ *Institutiones calculi integralis*, t. iii (2nd ed., 1793), pp. 381—475.

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The next achievement in progress is due to Legendre. Thus far, all the investigations had concentrated on the stationary requirement—the Euler test, when there is one independent variable; and the Lagrange test, when there are two independent variables. All of them were concerned with the terms of the first degree in the small quantities in the increment of the integral (the so-called 'first' variation), consequent upon a small increment of the independent variable. The customary requirements for a maximum or a minimum of an ordinary function, when the function is given explicitly, demand the consideration of the aggregate of terms of the second degree in those small quantities in its increment (the so-called 'second' variation). Legendre was the first to discuss this aggregate. To him is due the idea of modifying its initial expression, by adding suitable arbitrary variations within the range, and subtracting their corporate effect in the balance at the boundary, always subject to unaltered conditions at the boundary. This device enabled him to express the second variation in a compact form, the sign of which became of essential significance for the main purpose.

Legendre thus added* a further test, discriminating as to character, which must be satisfied by integrals in one independent variable if they are to possess a maximum or a minimum. The new requirement, like the Euler test and the Lagrange test, is a necessary qualifying test.

Jacobi.

6. The next name to be mentioned is that of Jacobi. Discussing the second variation as formulated by Legendre, he proved that a quantity, of critical importance in Legendre's expression, could always (without any inverse process) be derived from the knowledge presumed as attained in the Legendre stage. Making this advance in the purely mathematical range of the problem, he obtained extended forms for the tests already established; he constructed an essential modification of Legendre's form; and from this modified form, he deduced† a further test, limiting the range of the integral.

A full proof, and an extension (also with full proof), of Jacobi's results were given in a memoir by Hesse‡; and further investigations connected with the transformation of the second variation are due to Clebsch§, and to Mayer‖, though their inferences for the most part are left without a statement of the geometrical significance of the form of the second variation, such as was made by Jacobi.

The character of Jacobi's test admits of simple illustration. The shortest arc on the surface of a sphere, between two points on that surface, is the

* *Mémoires de l'Académie Royale des Sciences* (1786), pp. 7—37.

† *Crelle's Journal*, t. xvii (1837), pp. 68—82.

‡ *Crelle's Journal*, t. liv (1857), pp. 227—273.

§ *Crelle's Journal*, t. lv (1858), pp. 254—273, 335—355.

‖ *Beiträge zur Theorie der Maxima und Minima einfacher Integrale* (Leipzig, 1866); *Crelle's Journal*, t. lxix (1868), pp. 238—263.

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portion of the great circle passing through them and lying between them: so far, the tests of Euler and Legendre are satisfied. But the portion of the great circle must be less than half the circle; if it is a half-circle, the length (minimal in this case) is not unique; if it is greater than a half-circle, the length is certainly not a minimum between its extreme points, because (among other reasons) it can always be reduced by small variations. The requirement in this instance (and the cognate requirement in any corresponding problem) comes into Jacobi's work: the limit of the integral, as ranging between 'conjugate' points on the curve obtained, is bounded by a point on the circle and the diametrically opposite point.

The general test devised by Jacobi indicates the limit (if there be a limit) to the range along the characteristic curve within which the magnitude of the integral under consideration is a maximum or a minimum.

With the achievement of these results, essential progress in the solution of the problem again came to an end for a long period. Mathematical amplifications, and generalisations wider in range but cognate in character, were made from time to time; but, in real effect, there were only the three types of test, to be associated initially with the names of Euler and Lagrange, Legendre, and Jacobi.

General notion of variations.

7. In all the analysis hitherto applied, whatever its form, the general notion behind the mathematical expression was the same. A hypothetical maximum or a hypothetical minimum was postulated; if it were a maximum, every small change (called a *variation*) must lead to a decrease in the hypothetical maximum value; if it were a minimum, every such variation must lead to an increase in the hypothetical minimum value. Much explicit virtue lay in the assumption of the smallness of the variation. Not a little implicit difficulty was involved (though not recognised) in the lack of adequate specification of the properties constituting that smallness. Latent limitations (also not recognised) were imposed by the use of the symbol δ, which was initially employed to indicate the actual small variation of the unknown magnitude and which, by argument presumably deemed too obvious for statement, was allowed to exercise a supposed and unchallenged parsimonious influence solely by its literal character, wherever and whenever it was made to appear. When once the range, within which the tacit assumptions are legitimate, is actually examined, the result is to shew that the small variations are themselves gravely limited in character. Other small variations can be postulated, initially the same as regards type of magnitude, but with action gravely different from the restricted influence exercised by the other magnitudes, which are derived from them by some of the operations that were freely used. One consequence is that the old tests, when satisfied, lead to inferences which are legitimate only within the range of the small

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variations that remain small, not only in their own influence, but through the influence to be exercised by these derivatives. Another consequence is that the old tests can be trusted solely as connected with such range of variation: and that, precisely because they cease to be applicable when the hypotheses leading to their establishment are no longer valid, these old tests can furnish no information for small variations of the other kind indicated. They will remain, as necessary tests, because the maximum or the minimum property must be possessed for all small variations and therefore for variations of the limited character. They are inadequate for the discussion of the magnitude, when it is subjected to small variations not of this limited character. In fact, the three tests are necessary for the solution of the problem; they are not sufficient to secure the completeness of the solution.

8. Further, another limitation was imposed upon the small variations adopted. It was imposed explicitly; no attempt was made to shew that it is immaterial; and any significance, that the limitation might have borne, was so far ignored as not even to come within the range of examination. The limitation consisted in restricting the variations so that they should affect the dependent variable alone and should not apply to the independent variable, either independently or concurrently with the changes made in the dependent variable. If we turn to the graphical representation, whereby the relation between the two variables is illustrated by a plane curve, the small variations adopted consisted of a small displacement along the ordinate, coupled with a small displacement of the direction of the tangent; but small displacements, made sideways off the ordinate, were never entertained. Occasionally, such a displacement had to be considered for an isolated point or for isolated points; thus the extreme point might be required to lie on a given curve and be not merely a fixed point. In such instances, a special detailed argument was applied to take account of the effect caused by such a displacement of the particular point, while the possible effect of such a displacement throughout the range was ignored.

When the corresponding investigation, necessary to take these small variations into consideration (subject to the old limitations as to the character of the variation of their derivatives), has been completed, it is found that no new test emerges. The sole change is of a formal character; but such a result requires definite establishment. The net conclusion, however, is that, for small variations which are subject to the limitations indicated, three necessary critical tests exist.

Weak variations: strong variations.

9. Now these small variations, even when imposed upon both the dependent variable and the independent variable, are only the simplest type of all small variations. Thus, to return to the graphical representation, we could

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have small variations of a curve such as are given by displacements corresponding to very rapid small oscillations. In these, while the change of position is small, the change of direction may be finite though not small; the change of curvature may be very large; and so on. For all such variations, the analysis that has been effective in the other stages no longer applies; some new process must be devised that will cause these less simple variations to be taken into account.

Small variations of the earlier type, viz. *those in which the derivatives of the variation are of the same order of smallness as the variation itself*, are often called *weak* variations. Among weak variations, those which are restricted so that they affect the dependent variable alone, will be called *special* variations (the quality of weakness being tacitly included in the name). Small variations of the later type, viz. those in which the derivatives are not limited to be small, though the actual displacement is definitely small,—are called *strong* variations. Manifestly strong variations will be much more versatile in character than weak variations; and progress will initially be secured by considering selected types of strong variations.

Weierstrass.

10. An adequate method, new in form and (as regards later developments) new in substance, was initiated by the work of Weierstrass. Although he died as long ago as 1897, no fully authoritative exposition of his researches has yet appeared. He gave lectures on the subject in 1872 and 1879, perhaps earlier, in Berlin. Notes of his lectures have circulated, though without indications as to whether their range is complete or only partial. The work of Schwarz on minimal surfaces, as on branches of the theory of analytical functions, may be not unfairly presumed to have had its foundations in the teaching and the work of his master, whom he succeeded at Berlin. The results of the earlier stages of Weierstrass's work have found a partial record in some books* published since his death. Renewed interest was stirred in the subject owing to his influence; and writers, in varying degrees of independence, have been stimulated to further researches. Besides Schwarz, the names of Hilbert, Goursat, Hadamard may fitly be mentioned, as well as those of Kobb, Bolza, Kneser, Veblen, Hedrick, and Tonelli†. The work of Clebsch also was important, though it belongs to the older range; and Todhunter's two volumes, dealing with the history of that range, may be consulted with advantage.

Two contributions, different in kind yet connected, and both of funda-

* Hancock's *Calculus of Variations* may be consulted in this connection.

† A considerable amount of research has appeared in scattered articles in the *Bulletin of the American Mathematical Society*. An article, by Love, in the supplement to the *Encyclopædia Britannica*, gives a concise account of the earlier stages of the older theory; and a fuller account of the subject will be found in two articles, by Kneser, and by Zermelo and Hahn, in the *Encyclopädie d. math. Wissenschaft*, vol. ii, A. 1, pp. 571—625, 626—641.

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mental importance, are made by the work of Weierstrass. In one of these, he ignores the preferential selection of the dependent variable as the subject of variation; with him, the dependent variable and the independent variable are alike subjected to arbitrary small variations, independently of one another. The old tests, in modified amplification and with added elucidation, are obtained, in connection with weak variations. The discussion of strong variations, and the deduction of one new test—the so-called Excess-function—derived from the consideration of one simple class of strong variations, are entirely due to his initial researches in the matter. But he dealt with only the simplest class of strong variations, the particular class being sufficient for the purpose of the particular type of problem which he had under consideration. It may be that similar researches, involving less restricted strong variations for less particular types of problem, will only lead to generalisations, however important, of his Excess-function; it would appear as if such researches still remained for investigation.

11. As regards the weak variations, a further remark may be made to confirm the desirability of examining variations other than those which have been entitled 'special.' The more immediate geometrical representation, of the relation between a dependent variable and an independent variable by means of a plane curve, would imply one kind of special variation, were the analytical equivalent expressed by Cartesian coordinates; it would imply a different kind of special variation, were that equivalent represented by the usual polar coordinates; for homogeneous coordinates, and for bipolar coordinates, other different kinds of special variation would be used. All possible kinds of special variations are included when the most general kind of weak variation is adopted, because each special variation is only one particular case of the general type.

Moreover, the general variation includes, as a particular case, that apparent variation which consists of displacement merely along the length of a curve without any alteration of its shape. Such an apparent variation is not a variation of the value of the integral because, when the integral is regarded as a sum of infinitesimal elements between its upper limit and its lower limit, the value of the integral is independent of the precise detailed mode of selection of its constituent elements.

Lastly, the use of general variations makes it possible to consider (without the supplementary discussions needed for special variations) the cases, when the limits of the integral are liable to partial (though not quite arbitrary) mobility instead of being definitely fixed.

Assumed limitations on functionality.

12. Throughout all the investigations, we shall suppose, and now explicitly assume, that the subject of integration either (i) is a regular function* of its

* That is, a function which (with its derivatives) is uniform, finite and continuous.

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arguments within the range of integration: or (ii), when not a uniform function for all possible values of its arguments within the whole field of variation, remains a uniform branch of some function, so that it behaves as a regular function in each part of the range of integration.

If, at any place or places, an argument or arguments should suffer discontinuity in value, so that the function or any derivative of the function potentially may cease to be regular, explicit note of the fact or of the possibility will be taken. Again, during the analysis and unless a warning to a contrary effect is given, the solutions of ancillary differential equations will be regarded as providing regular functions, when these are determined by regular data. In particular instances, it may happen that the conditions, which are imposed, cannot be satisfied by regular functions; it may even happen that, of a postulated problem, no solution exists of a type which is regular: the impossibility will be noted. But we shall proceed on an initial assumption that a regular solution may exist and shall make its discovery our quest. The hypothetical solution of that type (perhaps with the assistance of existence-theorems, established with similar initial assumptions) will be subjected to all the conditions, and will be made to conform to all the data, that are imposed and provided. When the conditions are satisfied and the data are incorporated, the result will be regarded as the solution. In the contrary event, note will be taken of the fact, which thence will have emerged, that the imposed conditions and the assigned data forbid a solution of the foregoing type. It may be that conditions could be modified and that data could be changed; such alterations could be considered in their turn, as raising a new specific problem.

Moreover, unless some contrary demand occurs requiring satisfaction, we shall expect that the functions concerned are of an ordinary simple character, using these words as in common parlance. Unless expressly forbidden, we shall suppose that our functions have derivatives which are unique, save at isolated singularities or in the immediate vicinity of singularities; *e.g.*, we shall not be concerned with uniform continuous functions, which nowhere have a unique differential coefficient. Equally we shall suppose that all the functions, with which we have to deal, are capable of unconditional integration, *e.g.*, that they are not beset with singularities. The present purpose is the development of a calculus, without delaying over the foundations of Analysis, which will be regarded as having been duly laid and recognised in any bearing that has significance. It may happen that, when strong variations are considered in a full detail which would be essential in a complete discussion of their influence, corresponding refinements will prove necessary. For the present, its more ordinary and less critical commonplace range will be held sufficient; and, of course, the functions admitted will be subject to the corresponding limitations as regards regularity, continuity, and variation.

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